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# Euler equation on a fast rotating sphere—Time-averages and zonal flows

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ABSTRACT

Motivated by recent studies in geophysical and planetary sciences, we investigate the PDE-analytical aspects of time-averages for barotropic, inviscid flows on a fast rotating sphere $\mathbb{S}^2$. Of particular interest is the incompressible Euler equation. We prove that finite-time-averages of solutions stay close to a subspace of *longitude-independent zonal flows*. The initial data are unprepared and can be arbitrarily far away from this subspace. Our analytical study justifies the global Coriolis effect in the spherical geometry as the underlying mechanism of this phenomenon. We use Riemannian geometric tools including the Hodge theory in the proofs.



## 1. Introduction

Recent studies have seen increasing understandings of *global* characteristics of geophysical flows on Earth and giant planets in the Solar System. Simulations and observations have persistently shown that coherent anisotropy favoring *zonal flows* appears ubiquitously in planet scale circulations. For a partial list of computational results, we mention [1] for 3D models, [2–6] for 2D models, and references therein. These highly resolved, eddy-permitting simulations are made possible by rapid developments of high performance computing. On the other hand, we have observed zonal flow patterns (bands and jets) on giant planets for hundreds of years, which has attracted considerable interests recently thanks to spacecraft missions and the launch of the Hubble Space Telescope (e.g. [7,8]). Fig. 1.1 shows a composite view of the banded structure of Jovian atmosphere captured by the Cassini spacecraft [9]. There are also observational data in the oceans on Earth showing persistent zonal flow patterns (e.g. [10–12]).

The geophysical literature above motivates our mathematical study of zonal flows on a fast rotating sphere in the context of global dynamics. Our analysis verifies that zonal flow patterns in time-averages are sustained by the spherical geometry and the associated large meridional gradient of the Coriolis parameter. Note that our analysis is performed on the entire sphere instead of a local patch of the sphere. Also note that, in the aforementioned literature of numerical results, it is necessary to implement the north–south variation of Coriolis effects and to apply time-averaging to the data in order for zonal flow patterns to emerge. These two key ingredients are both included in our proofs.

To this end, we study inviscid, barotropic geophysical flows on a unit sphere $\mathbb{S}^2$ centered at the origin of $\mathbb{R}^3$ and fast rotating about the $z$-axis with constant angular velocity. Let vector field $\mathbf{u}(t,q)$, tangent to $\mathbb{S}^2$ at every point $q \in \mathbb{S}^2$, denote the fluid velocity *relative* to this rotating frame. Throughout this paper, we represent any point $q \in \mathbb{S}^2$ either by its relative-to-the-frame Cartesian coordinates $(x, y, z)$ or its relative-to-the-frame spherical coordinates $(\theta, \phi)$ with $\theta$ being the colatitude and $\phi$ the longitude.[1] Also, let $\mathbf{e}_\phi$ be the unit vector in the zonal direction of increasing longitude and $\mathbf{e}_\theta$ be the unit vector in the meridional direction of increasing colatitude.

In this study, we focus on a canonical PDE system: the incompressible Euler equation under the Coriolis force [13–15],

$$\begin{aligned} &\partial_t \mathbf{u} + \nabla_{\mathbf{u}} \mathbf{u} + \nabla P = \frac{z}{\varepsilon} \mathbf{u}^{\perp}, \qquad \operatorname{div} \mathbf{u} = 0, \\ &\mathbf{u}(0, \cdot) = \mathbf{u}_0 \quad \text{with } \operatorname{div} \mathbf{u}_0 = 0 \end{aligned} \tag{1.1}$$

where constant $\varepsilon$, called the Rossby number, scales like the frequency of the frame's rotation (usually 0.01–0.1 at a global scale),

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[1] At $z = 1$, we fix $\theta = 0$ but $\phi$ can be arbitrary. Such singularity issue does not occur for Cartesian coordinates $x, y, z \in C^\infty(\mathbb{S}^2)$.

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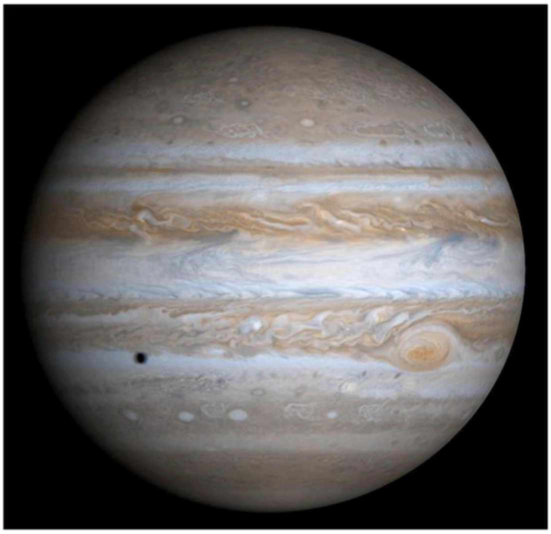

**Fig. 1.1.** This true-color simulated view of Jupiter is composed of 4 images taken by NASA's Cassini spacecraft on December 7, 2000. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)
*Source:* NASA/JPL/University of Arizona [9].

and $\mathbf{u}^\perp$ denotes a counterclockwise $\pi/2$-rotation of $\mathbf{u}$ on $\mathbb{S}^2$. Cartesian coordinate $z$ indicates how the Coriolis parameter varies along the meridional direction. This means the Coriolis effect is not uniformly strong and actually vanishes on the equator. It is the large *gradient* of the Coriolis parameter that drives the zonal flow patterns.

The wellposedness of the above system and related ones has been well established in literature. Please refer to [16–18] and references therein for further discussion. In short, with initial data $\mathbf{u}_0$ in a Sobolev space $H^k(\mathbb{S}^2)$ for integer $k \geq 3$, the solution exists uniquely in a time interval that depends on $\varepsilon$ and the $H^k$ norm of $\mathbf{u}_0$.

Our theoretical investigation is then focused on the fast rotating regime with $\varepsilon \ll 1$ and the nature of the *time-averages* of $\mathbf{u}$:

$$\overline{\mathbf{u}}(T, \cdot) := \frac{1}{T}\int_0^T \mathbf{u}(t, \cdot)\, dt \tag{1.2}$$

for positive times $T$. Since we admit arbitrary, unprepared initial data without any coherent structure, it is necessary to use time-averaging (1.2) to extract the zonal flow pattern.

The main result is stated as the following.

**Theorem 1.1.** *Consider the incompressible Euler equation (1.1) on $\mathbb{S}^2$ with initial data $\mathbf{u}_0 \in H^k(\mathbb{S}^2)$ for $k \geq 3$. Define the time-averaged flow $\overline{\mathbf{u}}$ as in (1.2). Then, there exists a function $f(\cdot) : [-1, 1] \mapsto \mathbb{R}$ and constants $C_0, T_0$ independent of $\varepsilon$ and $\mathbf{u}_0$, s.t. for any given $T \in [0, T_0/\|\mathbf{u}_0\|_{H^k}]$,*

$$\left\|\overline{\mathbf{u}}(T, x, y, z) - \nabla^\perp f(z)\right\|_{H^{k-3}(\mathbb{S}^2)} \leq C_0 \varepsilon \left(\frac{M_0}{T} + M_0^2\right), \tag{1.3}$$

*with $M_0 := \|\mathbf{u}_0\|_{H^k}$. In spherical coordinates, the approximation $\nabla^\perp f(z)$ is*

$$\nabla^\perp f(z) = -f'(\cos\theta)\sin\theta\, \mathbf{e}_\phi,$$

*which is a* longitude — independent zonal flow.

Our theoretical result proves computational and observational results in the literature mentioned at the beginning of this paper. In particular, we show that the zonal-flow pattern becomes increasingly prominent with decreasing Rossby number $\varepsilon \searrow 0$. The time-averaged flow $\overline{\mathbf{u}}$ is only $O(\varepsilon)$ away from a very restricted subspace consisting of longitude-independent zonal flows. The initial data, on the other hand, are unprepared and can be arbitrarily far away from that subspace of zonal flows. Our proofs below will suggest that such unique pattern is essentially due to the Coriolis parameter $z/\varepsilon$ that varies meridionally from the strongest at the poles to zero on the equator.

Rossby number $\varepsilon$ is typically at magnitude 0.01–0.1 for Earth oceans, which results in a time scale of magnitude 10–100 Earth days according to Theorem 1.1. This suggests that zonal flow patterns can occur at time scales below those used in the literature. In fact, the Rossby number is even smaller for giant planets, leading to the direct observability of banded structures.

The proof of our Main Theorem, formally speaking, starts with a reformulation of (1.1) using the Hodge decomposition (cf. Section 2) into

$$\partial_t \mathbf{u} + \nabla^\perp \Delta^{-1} \operatorname{curl}(\nabla_{\mathbf{u}} \mathbf{u}) = \frac{1}{\varepsilon} \nabla^\perp \Delta^{-1} \operatorname{curl}(z\mathbf{u}^\perp). \tag{1.4}$$

Then, the large constant $\frac{1}{\varepsilon}$ leads to an $O(\varepsilon)$ estimate on the *time-average* of the $\nabla^\perp\Delta^{-1}\operatorname{curl}(z\mathbf{u}^\perp)$ term and eventually leads to the Main Theorem. This procedure fits into the abstract framework of the following lemma.

**Lemma 1.1.** *Consider time-dependent equation,*

$$\partial_t \mathbf{u} = \frac{1}{\varepsilon}\mathcal{L}[\mathbf{u}] + f$$

*where $0 < \varepsilon \ll 1$ is a scaling constant, $\mathcal{L}$ is a linear operator and $f$ includes nonlinear and source terms. Assume* a priori *that, for some Hilbert spaces $\mathrm{X}_1, \mathrm{X}_2$, one has*

$$\mathbf{u} \in \mathcal{C}([0, T], \mathrm{X}_1 \cap \mathrm{X}_2), \qquad \mathcal{L}[\mathbf{u}] \in \mathcal{C}([0, T], \mathrm{X}_2), \qquad f \in \mathcal{C}([0, T], \mathrm{X}_2), \tag{1.5}$$

*and for any such solution,*

$$\int_0^T \mathcal{L}[\mathbf{u}]\, dt = \mathcal{L}\left[\int_0^T \mathbf{u}\, dt\right]. \tag{1.6}$$

*Also, let operator $\prod_{\mathrm{null}\{\mathcal{L}\}} : \mathrm{X}_1 \to \mathrm{X}_1$ denote (some) projection onto the kernel of $\mathcal{L}$.*

*Then, under the assumption*

$$\|\mathbf{u} - \prod_{\mathrm{null}\{\mathcal{L}\}} \mathbf{u}\|_{\mathrm{X}_1} \leq C\|\mathcal{L}[\mathbf{u}]\|_{\mathrm{X}_2} \tag{1.7}$$

*for some constant $C$, the following estimate holds true on the* time — average *of* $\mathbf{u}$,

$$\left\|\frac{1}{T}\int_0^T \mathbf{u}\, dt - \frac{1}{T}\int_0^T \prod_{\mathrm{null}\{\mathcal{L}\}} \mathbf{u}\, dt\right\|_{\mathrm{X}_1} \leq \varepsilon C\left(\frac{2M}{T} + M'\right)$$

*where constants $M := \max_{t\in[0,T]} \|\mathbf{u}(t, \cdot)\|_{\mathrm{X}_2}$ and $M' := \max_{t\in[0,T]} \|f(t, \cdot)\|_{\mathrm{X}_2}$.*

**Remark 1.1.** Estimate (1.7) is the key assumption and a majority of this article is related to its proof for the particular case of (1.4). Note that (1.7) is an elementary fact in a finite-dimensional space where $\mathbf{u}$ is a vector in $\mathbb{R}^n$, $\mathcal{L}$ a linear transform $\mathbb{R}^n \mapsto \mathbb{R}^n$ and $\prod_{\mathrm{null}\{\mathcal{L}\}}$ the $l^2$-projection onto $\mathrm{null}\{\mathcal{L}\}$. In such case, estimate (1.7), with the norms understood as $l^2$ norm on both sides, amounts to the boundedness of $\mathcal{L}^{-1} : \mathrm{image}\{\mathcal{L}\} \mapsto \mathbb{R}^n/\mathrm{null}\{\mathcal{L}\}$.

**Remark 1.2.** The idea of estimating time-averages for PDE systems with fast oscillations has appeared in e.g. [19, Th. 2.5], [20].

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**Proof of Lemma 1.1.** First, transform the original equation into

$$\mathbf{u}_t = \frac{1}{\varepsilon}\mathcal{L}\left[\mathbf{u} - \prod_{\text{null}\{\mathcal{L}\}} \mathbf{u}\right] + f$$

and apply time-averaging $\frac{1}{T}\int_0^T \cdot\, dt$ on both sides

$$\frac{1}{T}\left(\mathbf{u}(T,\cdot) - \mathbf{u}(0,\cdot)\right) = \frac{1}{\varepsilon T}\int_0^T \mathcal{L}\left[\mathbf{u} - \prod_{\text{null}\{\mathcal{L}\}} \mathbf{u}\right] dt + \frac{1}{T}\int_0^T f(t,\cdot)dt.$$

By (1.6), the above equation becomes

$$\frac{1}{T}\left(\mathbf{u}(T,\cdot) - \mathbf{u}(0,\cdot)\right) = \frac{1}{\varepsilon}\mathcal{L}\left[\frac{1}{T}\int_0^T \mathbf{u}\, dt - \frac{1}{T}\int_0^T \prod_{\text{null}\{\mathcal{L}\}} \mathbf{u}\, dt\right] + \frac{1}{T}\int_0^T f(t,\cdot)\, dt.$$

Due to the factor $\frac{1}{\varepsilon}$ in the first term on the RHS, we have

$$\left\|\mathcal{L}\left[\frac{1}{T}\int_0^T \mathbf{u}\, dt - \frac{1}{T}\int_0^T \prod_{\text{null}\{\mathcal{L}\}} \mathbf{u}\, dt\right]\right\|_{X_2} \leq \varepsilon\left(\frac{2M}{T} + M'\right).$$

Finally, apply estimate (1.7) to arrive at the conclusion. □

We will use this lemma to organize various results that contribute to the proof of Main Theorem 1.1, some of which are interesting on their own. In a nutshell, we will define operators $\mathcal{L}[\mathbf{u}] := \nabla^\perp \Delta^{-1} \operatorname{curl}(z\mathbf{u}^\perp)$ in Definition 2.1 and $\prod_{\text{null}\{\mathcal{L}\}}$ (cf. Lemma 3.2), prove the key estimate (1.7) in Theorem 4.1 and verify regularity assumptions (1.5) and (1.6) (cf. Section 6).

Note that, the constant $M$ used in the above lemma depends on size of the solution up until time $T$ and is *not* necessarily independent of $\varepsilon$. A priori estimates uniform in $\varepsilon$ are therefore in order. The proof requires new considerations that complement the existing energy methods, which will be explained in the proofs leading to Theorem 5.1 in Section 5.

For the rest of this article, we will use differential operators such as Laplacian and gradient on $\mathbb{S}^2$, and integral-related concepts such as the $L^2(\mathbb{S}^2)$ inner product and "integrating-by-parts" formulas. These objects are closely related to their counterparts in Euclidean space $\mathbb{R}^2$ with subtle differences; and they are the subject of standard Differential Geometry theory. So, we will briefly explain them in the Appendix with the belief that the reader can proceed comfortably with such an arrangement. We start Section 2 with describing a version of the Hodge decomposition in terms of velocity fields. An operator $\mathcal{L}$ that plays the same role as the $\mathcal{L}$ in Lemma 1.1 is defined by the end of Section 2. In Section 3, we characterize the null space of $\mathcal{L}$, identifying null$\{\mathcal{L}\}$ as the space of longitude-independent zonal flows (cf. Lemma 3.1). We also define the projection operator $\prod_{\text{null}\{\mathcal{L}\}}$ and its complement. In Section 4, we obtain Sobolev-type estimates, in particular (1.7), regarding $\mathcal{L}$ and $\prod_{\text{null}\{\mathcal{L}\}}$ using the spherical harmonics. In Section 5, we prove solution regularity and $\varepsilon$-independent estimates upon carefully examining commutativity properties of some differential–integral operators on a sphere. Finally in Section 6, we sum up all these results to prove the Main Theorem 1.1. In the Appendix, we give the rigorous definitions of differential operators on surfaces such as $\mathbb{S}^2$ and prove important properties that are relevant here. It is necessary to adopt coordinate-independent differential geometric tools since any global coordinate system on $\mathbb{S}^2$ is bound to have singularity issues. On the other hand, one can formally use spherical coordinates as well as Cartesian coordinates for most of the arguments presented in this paper, knowing their validity is justified.

## 2. Hodge decomposition

The *Hodge decomposition* theorem [21,22], as generalizaion of the Helmholtz decomposition in $\mathbb{R}^3$, confirms that for any $k$-form $\omega$ on an oriented compact Riemannian manifold, there exist a $(k-1)$-form $\alpha$, $(k+1)$-form $\beta$ and a harmonic $k$-form $\gamma$, s.t.

$$\omega = d\alpha + \delta\beta + \gamma.$$

In particular, if the manifold is a surface in the cohomology class of $\mathbb{S}^2$ (loosely speaking, there is no "hole" or "handle"), then for any smooth vector field[2] $\mathbf{u}$, there exist two scalar-valued functions $\Phi$ (called potential) and $\Psi$ (called stream function) such that

$$\mathbf{u} = \mathbf{u}_{\text{irr}} + \mathbf{u}_{\text{inc}} \quad \text{where } \mathbf{u}_{\text{irr}} := \nabla\Phi \text{ and } \mathbf{u}_{\text{inc}} := \nabla^\perp\Psi, \tag{2.1}$$

where subscript "irr" stands for irrotational vector fields and "inc" for incompressible. Please refer to (A.15) in the Appendix and the discussion that leads to it. Moreover, the decomposition satisfies

$$\operatorname{curl}\mathbf{u}_{\text{irr}} = \operatorname{div}\mathbf{u}_{\text{inc}} = 0, \quad \text{and} \quad \mathbf{u}_{\text{irr}} = \nabla\Delta_{\mathbb{S}^2}^{-1}\operatorname{div}\mathbf{u},$$
$$\mathbf{u}_{\text{inc}} = \nabla^\perp\Delta_{\mathbb{S}^2}^{-1}\operatorname{curl}\mathbf{u}. \tag{2.2}$$

Such a decomposition is unique because a harmonic scalar-valued function on a sphere (and any surface in the same cohomology class) is always constant and therefore $\Delta_{\mathbb{S}^2}^{-1}$ is unique up to a constant. Note that this decomposition is actually valid for any square-integrable vector field on $\mathbb{S}^2$ by the virtue that smooth functions are dense in $L^2$. For simplicity, we will use $\Delta$ for $\Delta_{\mathbb{S}^2}$ from here on. Also, we assume that, unless specified otherwise,

$$\Delta^{-1}f \quad \text{always has zero global mean over } \mathbb{S}^2. \tag{2.3}$$

We will include the differential-geometric definitions and properties of $\nabla$, $\nabla^\perp$, curl, div, $\Delta$ on $\mathbb{S}^2$ in the Appendix. Many of them are the same as their counterparts in $\mathbb{R}^2$ *as long as* one does not invoke partial derivatives in local coordinates.

Now, rearrange (1.1) as

$$\frac{z}{\varepsilon}\mathbf{u}^\perp - \nabla_{\mathbf{u}}\mathbf{u} = \partial_t\mathbf{u} + \nabla P.$$

Observe that on the RHS, $\partial_t\mathbf{u}$ is incompressible and $\nabla P$ is irrotational. Thus, the RHS is the unique Hodge decomposition of the LHS, which satisfies the elliptic PDEs (2.2). In particular, the incompressible part $\partial_t\mathbf{u}$ is uniquely determined by

$$\partial_t\mathbf{u} = \nabla^\perp\Delta^{-1}\operatorname{curl}\left(\frac{z}{\varepsilon}\mathbf{u}^\perp - \nabla_{\mathbf{u}}\mathbf{u}\right). \tag{2.4}$$

In the context of Lemma 1.1, we define the following operator.

**Definition 2.1.** For any (square-integrable) vector field $\mathbf{u}$ on $\mathbb{S}^2$, not necessarily div-free, define

$$\mathcal{L}[\mathbf{u}] := \nabla^\perp\Delta^{-1}\operatorname{curl}(z\mathbf{u}^\perp). \tag{2.5}$$

Here, $\Delta^{-1}$ follows the convention (2.3).

Then, (2.4) can be reformulated as,

$$\partial_t\mathbf{u} + \nabla^\perp\Delta^{-1}\operatorname{curl}(\nabla_{\mathbf{u}}\mathbf{u}) = \frac{1}{\varepsilon}\mathcal{L}[\mathbf{u}]. \tag{2.6}$$

This formulation will be used for the rest of this article.

[2] We follow the convention that velocity field and vector field can be treated as the same mathematical object, so long as physical units are not of concern.

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## 3. Null space of $\mathcal{L}$ and associated $L^2$-orthogonal projection

We observe that the definition (2.5) naturally implies a sufficient condition for $\mathcal{L}[\mathbf{u}] = 0$ is

$$\text{curl}\,(z\mathbf{u}^\perp) = 0.$$

Indeed this is also necessary by the virtue of (A.12), namely $\text{curl}\,\nabla^\perp = \Delta$, so that

$$\text{curl}\,\mathcal{L}[\mathbf{u}] = \text{curl}\,\nabla^\perp\Delta^{-1}\,\text{curl}\,(z\mathbf{u}^\perp) = \text{curl}\,(z\mathbf{u}^\perp).$$

Thus, for velocity field $\mathbf{u}$, not necessarily div-free,

$$\mathcal{L}[\mathbf{u}] = 0 \iff \text{curl}\,(z\mathbf{u}^\perp) = 0. \tag{3.1}$$

Further analysis reveals the following lemma.

**Lemma 3.1** (*Characterization of* null$\{\mathcal{L}\}$). *For div-free* $\mathbf{u}$ *with sufficient regularity,*

$$\mathcal{L}[\mathbf{u}] = 0 \iff \mathbf{u} = \nabla^\perp g(z) = -g'(\cos\theta)\sin\theta\,\mathbf{e}_\phi \tag{3.2}$$

*for some function* $g : [-1, 1] \mapsto \mathbb{R}$. *Thus, we identify* null$\{\mathcal{L}\}$, *when restricted to div-free velocity fields, with the space of* longitude — independent zonal flows.

**Proof of Lemma 3.1.** Apply the product rule (A.17) in the Appendix to $\text{curl}\,(z\mathbf{u}^\perp)$,

$$\text{curl}\,(z\mathbf{u}^\perp) = (\nabla z)\cdot\mathbf{u} + z\,\text{div}\,\mathbf{u} = (\nabla z)\cdot\mathbf{u} \tag{3.3}$$

where the $\cdot$ product is given by the natural metric on $\mathbb{S}^2$ induced from $\mathbb{R}^3$ and we used the incompressibility condition $\text{div}\,\mathbf{u} = 0$. Therefore, by (3.1) and (3.3), the null space of $\mathcal{L}$ is identified as

$$\text{For any div-free } \mathbf{u},\ \mathcal{L}[\mathbf{u}] = 0 \iff (\nabla z)\cdot\mathbf{u} = 0. \tag{3.4}$$

Since $\nabla z$ is in the meridional direction, (3.4) implies any $\mathbf{u}$ in the null space of $\mathcal{L}$ flows in the zonal direction. But there is more than that. Hodge decomposition (2.1), (2.2) implies

$$\text{div}\,\mathbf{u} = 0 \iff \mathbf{u} = \nabla^\perp\Psi(x, y, z) \tag{3.5}$$

with $\Psi$ being a unique scalar function with zero global mean. Combining (3.5) it with (3.4), we have, for any incompressible velocity field $\mathbf{u}$,

$$\mathbf{u} = \nabla^\perp\Psi \in \text{null}\{\mathcal{L}\} \iff (\nabla z) \parallel (\nabla\Psi).$$

The condition $(\nabla z) \parallel (\nabla\Psi)$ implies $\Psi$ is a function of $z$ only. Thus, we arrive at the conclusion. The very last term in (3.2) is due to the fact that, in spherical coordinates,

$$\nabla^\perp z = -\sin\theta\,\mathbf{e}_\phi. \quad \square$$

It is then easy to show the following characterization of $\prod_{\text{null}\{\mathcal{L}\}}$, the $L^2$-orthogonal-projection operator onto null$\{\mathcal{L}\}$.

**Lemma 3.2** (*Characterization of* $\prod_{\text{null}\{\mathcal{L}\}}$). *For any div-free vector field* $\mathbf{u} \in L^2(\mathbb{S}^2)$, *its* $L^2$-*orthogonal-projection onto* null$\{\mathcal{L}\}$ *satisfies*

$$\prod_{\text{null}\{\mathcal{L}\}}\mathbf{u} = \frac{\left(\oint_{C(\theta)}\mathbf{u}\cdot\mathbf{e}_\phi\right)\mathbf{e}_\phi}{\oint_{C(\theta)}\mathbf{e}_\phi\cdot\mathbf{e}_\phi} = \frac{1}{2\pi\sin\theta}\left(\oint_{C(\theta)}\mathbf{u}\cdot\mathbf{e}_\phi\right)\mathbf{e}_\phi \tag{3.6}$$

*where* $\oint_{C(\theta)}$ *is the line integral along the circle* $C(\theta)$ *at a fixed colatitude* $\theta$.

Several remarks are in order. First, among all possible projection operators, we chose one that nullifies the $L^2(\mathbb{S}^2)$ orthogonal complement of null$\{\mathcal{L}\}$. Secondly, intuitively, $\prod_{\text{null}\{\mathcal{L}\}}\mathbf{u}$ at a given latitude is a uniform zonal flow equal to the mean circulation of $\mathbf{u}$ at that latitude; thus, $(\text{id} - \prod_{\text{null}\{\mathcal{L}\}})\mathbf{u}$ is of zero circulation along the circle at a fixed latitude. Such intuition is consistent with the orthogonality condition

$$\int_{\mathbb{S}^2}\left(\prod_{\text{null}\{\mathcal{L}\}}\mathbf{u}\right)\cdot\left(\text{id} - \prod_{\text{null}\{\mathcal{L}\}}\right)\mathbf{u}' = 0.$$

Thirdly, even though (3.6) runs into singularity at the poles $\theta = 0$ and $\theta = \pi$, such singularity is removable. In fact, apply the Stokes' theorem on the RHS of (3.6) so that, for $\theta \in (0, \pi)$,

$$\prod_{\text{null}\{\mathcal{L}\}}\mathbf{u} = \frac{1}{2\pi\sin\theta}\left(\iint_{\text{interior of } C(\theta)}\text{curl}\,\mathbf{u}\right)\mathbf{e}_\phi$$

and by taking the limit as $\theta \to 0+$ and $\theta \to \pi-$, we obtain

$$\prod_{\text{null}\{\mathcal{L}\}}\mathbf{u}|_{poles} = \mathbf{0}.$$

In terms of the stream function, for any div-free velocity field $\mathbf{u} = \nabla^\perp\Psi$ which amounts to $\mathbf{u} = (\partial_\theta\Psi)\mathbf{e}_\theta{}^\perp + \left(\frac{\partial_\phi\Psi}{\sin\theta}\right)\mathbf{e}_\phi{}^\perp$ with removable singularity at the poles,

$$\prod_{\text{null}\{\mathcal{L}\}}(\nabla^\perp\Psi) = \frac{1}{2\pi}\left(\int_0^{2\pi}\partial_\theta\Psi\,d\phi\right)\mathbf{e}_\phi$$
$$= \nabla^\perp\left(\frac{1}{2\pi}\int_0^{2\pi}\Psi\,d\phi\right). \tag{3.7}$$

In other words, $\prod_{\text{null}\{\mathcal{L}\}}$ maps the stream function to its zonal means.

## 4. Key estimates

This section is dedicated to proving an estimate in the same spirit as (1.7). We will need the tools of Sobolev norms and inner product associated with the Hilbert space $L^2(\mathbb{S}^2)$. The definition and properties of $L^2$ inner product are given later in Appendix A.3 in terms of differential geometry. In this section, it suffices to accept that, for scalar fields $f, g$, their $L^2$ inner product $\langle f, g\rangle_{L^2(\mathbb{S}^2)} := \int_{\mathbb{S}^2}\bar{f}g\,d\Omega = \int_0^\pi\int_0^{2\pi}\bar{f}g\sin\theta\,d\theta d\phi$ with $\bar{f}$ being the complex conjugate of $f$ and $d\Omega$ being the area element of $\mathbb{S}^2$. We also follow the convention that, for vector fields $\mathbf{u}$, $\mathbf{v}$, $\langle\mathbf{u}, \mathbf{v}\rangle_{L^2(\mathbb{S}^2)} := \int_{\mathbb{S}^2}\bar{\mathbf{u}}\cdot\mathbf{v}\,d\Omega$ with the dot-product taken in the geometry of $\mathbb{R}^3$.

In this context, a convenient tool in studying Sobolev norms and $L^2$ inner products on $\mathbb{S}^2$ is the spherical harmonics. Define the spherical harmonic of degree $l$ and order $m$,

$$Y_l^m(\phi, \theta) := N_l^m e^{im\phi}Q_l^m(\cos\theta), \quad \text{for } l = 0, 1, 2, \ldots$$
$$\text{and } m = -l, -l+1, \ldots, 0, \ldots, l-1, l$$

where the normalizing constant $N_l^m = \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}}$ so that $\|Y_l^m\|_{L^2(\mathbb{S}^2)} = 1$. It satisfies the eigenvalue problem

$$\Delta Y_l^m = -l(l+1)Y_l^m \tag{4.1}$$

and orthonormal condition

$$\langle Y_l^m, Y_{l'}^{m'}\rangle_{L^2(\mathbb{S}^2)} = \delta_{ll'}\delta_{mm'}. \tag{4.2}$$

The associated Legendre polynomial $Q_l^m(z)$ satisfies the general Legendre equation

$$\frac{d}{dz}\left(\sqrt{1-z^2}\frac{d}{dz}Q_l^m(z)\right) + \left(l(l+1) - \frac{m^2}{1-z^2}\right)Q_l^m = 0$$

and can be expressed via the Rodrigues' formula, $Q_l^m(z) = \frac{1}{2^l l!}(1-z^2)^{\frac{m}{2}} \frac{d^{m+l}}{dz^{m+l}}(1-z^2)^l$.

By (4.1), (4.2) and the $L^2$-completeness of spherical harmonics, one obtains the spherical-harmonic series expansion of a scalar function $\Psi$ with $\int_{\mathbb{S}^2} \Psi = 0$

$$\Psi = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \psi_l^m Y_l^m, \quad \text{where } \psi_l^m = \langle \Psi, Y_l^m \rangle_{L^2(\mathbb{S}^2)}, \tag{4.3}$$

and the corresponding spherical-harmonic version of Parseval's identity

$$\|\Psi\|_{L^2(\mathbb{S}^2)} = \sqrt{\sum_{l=1}^{\infty} \sum_{m=-l}^{l} |\psi_l^m|^2}. \tag{4.4}$$

**Remark 4.1.** Here and below, we assume $\psi_0^0 = \int_{\mathbb{S}^2} \Psi = 0$ and exclude $l = 0$ from any series.

In order to estimate the Sobolev norms (esp. $H^k$ norms) of $Y_l^m$, we take the $L^2(\mathbb{S}^2)$ inner product of (4.1) with $Y$ (omitting indices for simplicity), invoke Green's identities (A.21) to calculate

$$\begin{aligned} l(l+1) &= l(l+1)\langle Y, Y\rangle_{L^2(\mathbb{S}^2)} \\ &= -\langle \Delta Y, Y\rangle_{L^2(\mathbb{S}^2)} = \langle \nabla Y, \nabla Y\rangle_{L^2(\mathbb{S}^2)} \end{aligned}$$
$$\Longrightarrow Y \in H^1(\mathbb{S}^2), \quad \text{and inductively,} \quad Y \in H^k(\mathbb{S}^2), \quad k \geq 0.$$

As a matter of fact, a little more rigor is needed in defining Sobolev norms on a manifold (e.g. [22]), in part due to the requirement that these definitions should be independent of coordinate systems; and their exact formulas may vary through the literature. Among many equivalent definitions, the following one is inspired by its analogue[3] in $\mathbb{R}^2$ and Green's identities (A.21),

$$\|f\|_{H^k(\mathbb{S}^2)} \approx \sqrt{\sum_{j=0}^{k} (-1)^j \langle \Delta^j f, f\rangle_{L^2(\mathbb{S}^2)}}.$$

Here, $A \approx B$ means $A/B$ is bounded uniformly from above and below by positive constants that only depend on $k$. Then, by (4.1), (4.2), one has

$$\|Y_l^m\|_{H^k(\mathbb{S}^2)} \approx \sqrt{\sum_{j=0}^{k} (l^2+l)^j} \approx (1+l)^k,$$

and certain orthogonality condition,

$$\sum_{j=0}^{k} (-1)^j \langle \Delta^j Y_l^m, Y_{l'}^{m'}\rangle_{L^2(\mathbb{S}^2)} = 0, \quad \text{if } (l,m) \neq (l',m').$$

Using the above three relations, we are now ready to introduce the following definition, extending the Parseval's identity (4.4) to $H^k$ norms.

**Definition 4.1.** For a scalar function $\Psi$ on $\mathbb{S}^2$ with $\int_{\mathbb{S}^2} \Psi = 0$ and series expansion (4.3), define its $H^k$ norm, among other equivalent versions, as

$$\|\Psi\|_{H^k} := \sqrt{\sum_{l=1}^{\infty} \sum_{m=-l}^{l} (1+l)^{2k} |\psi_l^m|^2}. \tag{4.5}$$

Consequently, we define $H^k$ norms for $\mathbf{u}$.

[3] i.e. $\|f\|_{H^k(\mathbb{R}^2)} \approx \sqrt{\sum_{j=0}^{k} \langle \Delta^{j/2} f, \Delta^{j/2} f\rangle_{L^2(\mathbb{R}^2)}} = \sqrt{\sum_{j=0}^{k} (-1)^j \langle \Delta^j f, f\rangle_{L^2(\mathbb{R}^2)}}$.

**Definition 4.2.** For a vector field $\mathbf{u}$ with Hodge Decomposition

$$\mathbf{u} = \nabla\Phi + \nabla^\perp\Psi \quad \text{with} \quad \int_{\mathbb{S}^2} \Phi = \int_{\mathbb{S}^2} \Psi = 0,$$

we define its $H^k$ norm, among other equivalent versions, as

$$\|\mathbf{u}\|_{H^k} := \sqrt{\|\Phi\|_{H^{k+1}}^2 + \|\Psi\|_{H^{k+1}}^2}. \tag{4.6}$$

In particular, if $\mathbf{u}$ is div-free with $\mathbf{u} = \nabla^\perp\Psi$ and $\int_{\mathbb{S}^2} \Psi = 0$, then

$$\left\|\nabla^\perp\Psi\right\|_{H^k} = \sqrt{\sum_{l=1}^{\infty} \sum_{m=-l}^{l} (1+l)^{2(k+1)} |\psi_l^m|^2}. \tag{4.7}$$

**Remark 4.2.** Here, we follow the zero mean convention in Remark 4.1, such that the above definition is consistent with $\|\mathbf{0}\|_{H^k} = 0$.

We now characterize operator $\mathcal{L}$ using the spherical harmonics. Consider incompressible velocity field $\mathbf{u} = \nabla^\perp\Psi$. By definition (2.5) and identity (3.3),

$$\mathcal{L}[\nabla^\perp\Psi] = \nabla^\perp\Delta^{-1}\left(\nabla z \cdot \nabla^\perp\Psi\right).$$

It is easy to verify that, in spherical coordinates,

$$\nabla z = -\sin\theta\, \mathbf{e}_\theta \quad \text{and} \quad \nabla^\perp\Psi = (\partial_\theta\Psi)\mathbf{e}_\theta{}^\perp + \left(\frac{\partial_\phi\Psi}{\sin\theta}\right)\mathbf{e}_\phi{}^\perp.$$

Thus, combining the three equalities above, we obtain

$$\mathcal{L}[\nabla^\perp\Psi] = \nabla^\perp\Delta^{-1}\partial_\phi\Psi. \tag{4.8}$$

**Lemma 4.1** (*Spherical-Harmonic Representation of $\mathcal{L}$*)**.** *For a scalar function $\Psi$ with a series expansion (4.3), the identity (4.8) leads to*

$$\mathcal{L}[\nabla^\perp\Psi] = \nabla^\perp \sum_{l=1}^{\infty} \sum_{\substack{m=-l \\ m\neq 0}}^{l} \frac{-im}{l(l+1)} \psi_l^m Y_l^m. \tag{4.9}$$

Here, we used the fact that $\partial_\phi Y_l^m = imY_l^m$ and $\Delta^{-1}Y_l^m = -\frac{1}{l(l+1)}Y_l^m$ for $l \geq 1$. Note that, following Remark 4.1, we exclude $l = 0$ from the series; we also exclude $m = 0$ since it doesn't contribute to (4.9) anyway.

We now use spherical harmonics to characterize the projection operator $\prod_{\text{null}\{\mathcal{L}\}}$ given in (3.7). It follows from (4.9) that

$$\mathbf{u} = \nabla^\perp\Psi \in \text{null}\{\mathcal{L}\} \iff \mathbf{u} = \nabla^\perp \sum_{l=1}^{\infty} \psi_l^0 Y_l^0,$$

which is consistent with Lemma 3.1 since $Y_l^0$ is a function of $\theta$ only. Therefore, the only modes that survive $\prod_{\text{null}\{\mathcal{L}\}}$ are those with $m = 0$ since $\prod_{\text{null}\{\mathcal{L}\}}$ is an $L^2$-orthogonal projection and by (A.21)

$$\langle \nabla^\perp Y_l^m, \nabla^\perp Y_{l'}^{m'}\rangle = -\langle \Delta Y_l^m, Y_{l'}^{m'}\rangle = l(l+1)\delta_{ll'}\delta_{mm'}.$$

**Lemma 4.2** (*Spherical-Harmonic Representation of $\prod_{\text{null}\{\mathcal{L}\}}$*)**.** *For a scalar function $\Psi$ with a series expansion (4.3),*

$$\prod_{\text{null}\{\mathcal{L}\}} \leq (\nabla^\perp\Psi) = \nabla^\perp\left(\sum_{l=1}^{\infty} \psi_l^0 Y_l^0\right) \tag{4.10}$$

$$\left(\text{id} - \prod_{\text{null}\{\mathcal{L}\}}\right)(\nabla^\perp\Psi) = \nabla^\perp\left(\sum_{l=1}^{\infty} \sum_{\substack{m=-l \\ m\neq 0}}^{l} \psi_l^m Y_l^m\right). \tag{4.11}$$

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Note that the above 2 equations can also be derived from (3.7) together with the fact that

$$\frac{1}{2\pi}\int_0^{2\pi} Y_l^m(\phi,\theta)d\phi = \delta_{m0}.$$

Combining (4.9) with (4.11) and using the absence of $m = 0$ modes from both series, we deduce that, when $\mathcal{L}$ is restricted to the image of $(\mathrm{id} - \prod_{\mathrm{null}\{\mathcal{L}\}})$, its null space is trivial and its inverse is "bounded" (as noted in Remark 1.1, this is automatically true for linear transform $\mathcal{L}: \mathbb{R}^n \mapsto \mathbb{R}^n$) in the following sense.

**Theorem 4.1.** *For any div-free vector field* $\mathbf{u} \in H^k(\mathbb{S}^2)$ *and* $k \geq 0$,

$$\left\| \mathbf{u} - \prod_{\mathrm{null}\{\mathcal{L}\}} \mathbf{u} \right\|_{H^k} \leq \left\| \mathcal{L}\left[ \mathbf{u} - \prod_{\mathrm{null}\{\mathcal{L}\}} \mathbf{u} \right] \right\|_{H^{k+2}} = \|\mathcal{L}[\mathbf{u}]\|_{H^{k+2}}.$$

**Proof.** Consider the stream function $\Psi$ so that $\mathbf{u} = \nabla^\perp \Psi$. Combining (4.7) and (4.11), we obtain

$$\|\mathbf{u} - \prod_{\mathrm{null}\{\mathcal{L}\}} \mathbf{u}\|_{H^k} = \sqrt{\sum_{l=1}^{\infty} \sum_{\substack{m=-l \\ m\neq 0}}^{l} (1+l)^{2(k+1)} |\psi_l^m|^2}.$$

Combining (4.7) and (4.9), we obtain

$$\|\mathcal{L}[\mathbf{u}]\|_{H^{k+2}} = \sqrt{\sum_{l=1}^{\infty} \sum_{\substack{m=-l \\ m\neq 0}}^{l} (1+l)^{2(k+3)} \left| \frac{m\psi_l^m}{l(l+1)} \right|^2}.$$

The key observation here is that $m = 0$ modes are absent in both series; thus, by a simple inequality

$$(1+l)^{2(k+1)} \leq (1+l)^{2(k+3)} \left| \frac{m}{l(l+1)} \right|^2 \quad \text{for } l \geq 1,\ |m| \geq 1,$$

we arrive at the conclusion. □

## 5. Uniform estimates independent of $\varepsilon$

In this section, we use energy methods to prove local-in-time existence of classical solutions for the incompressible Euler equation *independent of* the Rossby number $\varepsilon$. Recall the Eq. (2.6),

$$\partial_t \mathbf{u} + \nabla^\perp \Delta^{-1} \operatorname{curl}(\nabla_{\mathbf{u}} \mathbf{u}) = \frac{1}{\varepsilon}\mathcal{L}[\mathbf{u}], \tag{5.1}$$

where operator $\mathcal{L}$, as in (2.5), is defined by

$$\mathcal{L}[\mathbf{u}] := \nabla^\perp \Delta^{-1} \operatorname{curl}(z\mathbf{u}^\perp). \tag{5.2}$$

The standard energy method (e.g. [23, Ch. 3]) can by employed to prove, given initial data $\mathbf{u}_0 \in H^3(\mathbb{S}^2)$, a classical $C^1$ solution exists uniquely for a time interval $t \in [0, T_\varepsilon]$ that *may* depend on $\varepsilon$. Basically, by the Sobolev embedding theorem on Riemannian manifolds [24, Ch. 2 §10]

$$|f|_{W^{1,\infty}(\mathbb{S}^2)} \leq C\|f\|_{H^3(\mathbb{S}^2)}, \tag{5.3}$$

and that $H^3(\mathbb{S}^2)$ is compactly embedded in $C^1(\mathbb{S}^2)$, it suffices to obtain an $H^k$ estimate ($k \geq 3$).

To prepare for the proofs in $H^k$ norms, let us first show that the physical energy, $\|\mathbf{u}\|_{L^2}$, is actually *conserved* by the dynamics,

$$\|\mathbf{u}(t,\cdot)\|_{L^2} = \|\mathbf{u}(0,\cdot)\|_{L^2} \tag{5.4}$$

as long as $\mathbf{u}(t,\cdot)$ remains in $C^1(\mathbb{S}^2)$.

Take the $L^2$ inner product of $\mathbf{u}$ with both sides of (5.1)

$$\langle \mathbf{u}, \partial_t \mathbf{u}\rangle = -\langle \mathbf{u}, \nabla^\perp \Delta^{-1} \operatorname{curl}(\nabla_{\mathbf{u}}\mathbf{u})\rangle + \frac{1}{\varepsilon}\langle \mathbf{u}, \mathcal{L}[\mathbf{u}]\rangle. \tag{5.5}$$

The LHS equals $\frac{1}{2}\partial_t\|\mathbf{u}\|_{L^2}^2$ and we want to show the RHS is zero. First, by (A.20), (A.21),

$$\langle \mathbf{u}, \mathcal{L}[\mathbf{u}]\rangle = \langle \mathbf{u}, \nabla^\perp\Delta^{-1}\operatorname{curl}(z\mathbf{u}^\perp)\rangle = \langle \nabla^\perp\Delta^{-1}\operatorname{curl}\mathbf{u}, z\mathbf{u}^\perp\rangle.$$

Use (2.2) to further simplify

$$\langle \mathbf{u}, \mathcal{L}[\mathbf{u}]\rangle = \langle \mathbf{u}, z\mathbf{u}^\perp\rangle = 0. \tag{5.6}$$

Thus, the large term with $\frac{1}{\varepsilon}$ in (5.5) vanishes. Furthermore, by replacing $\mathbf{u}$ with $\mathbf{u}+\mathbf{v}$ for incompressible $\mathbf{u}, \mathbf{v}$ in the above identity, we show that

$$\langle \mathbf{u}, \mathcal{L}[\mathbf{v}]\rangle = -\langle \mathcal{L}[\mathbf{u}], \mathbf{v}\rangle, \tag{5.7}$$

i.e., $\mathcal{L}$ is a skew-symmetric operator w.r.t. $L^2(\mathbb{S}^2)$ inner product for incompressible vector fields.

By a similar procedure, the tri-linear term in (5.5) can be simplified as

$$\langle \mathbf{u}, \nabla^\perp\Delta^{-1}\operatorname{curl}(\nabla_{\mathbf{u}}\mathbf{u})\rangle = \langle \mathbf{u}, \nabla_{\mathbf{u}}\mathbf{u}\rangle.$$

To show the RHS vanishes, we introduce a more general lemma, which will be also useful for estimating the $H^k$ norms of the nonlinear advection terms.

**Lemma 5.1.** *Consider vector fields* $\mathbf{u}, \mathbf{v}$ *and scalar field* $f$ *all in* $C^1(\mathbb{S}^2)$, *and* $\mathbf{u}, \mathbf{v}$ *both tangent to* $\mathbb{S}^2$. *Then,*

$$\operatorname{div}_{\mathbb{S}^2}\mathbf{u} = 0 \quad \textit{implies}\ \langle f, \nabla_{\mathbf{u}} f\rangle = \langle \mathbf{v}, \nabla_{\mathbf{u}}\mathbf{v}\rangle = 0.$$

**Proof.** We first prove the scalar case $\langle f, \nabla_{\mathbf{u}} f\rangle$ which is the same as

$$\langle f, \mathbf{u}\cdot\nabla f\rangle = 0.$$

By the product rule (A.17) and $\operatorname{div}\mathbf{u} = 0$, we have, $\mathbf{u}\cdot\nabla f = \operatorname{div}(\mathbf{u}f)$, and thus

$$\langle f, \mathbf{u}\cdot\nabla f\rangle = \langle f, \operatorname{div}(\mathbf{u}f)\rangle = -\langle \nabla f, \mathbf{u}f\rangle$$

due to (A.19). It is easy to see that the RHS equals $-\langle \mathbf{u}\cdot\nabla f, f\rangle$ which leads to $\langle f, \mathbf{u}\cdot\nabla f\rangle = -\langle \mathbf{u}\cdot\nabla f, f\rangle = 0$.

For the vector case $\langle \mathbf{v}, \nabla_{\mathbf{u}}\mathbf{v}\rangle = 0$, invoke the definition of the covariant derivative (A.2) and the fact that $\mathbf{v}$ is tangent to $\mathbb{S}^2$ to obtain

$$\mathbf{v}\cdot(\nabla_{\mathbf{u}}\mathbf{v}) = \sum_{i=1}^{3} v_i\mathbf{e}_i \cdot \left(\sum_{i=1}^{3}(\nabla_{\mathbf{u}} v_i)\mathbf{e}_i\right) = \sum_{i=1}^{3} v_i\nabla_{\mathbf{u}} v_i.$$

Since each $v_i$ is a scalar field, by the first part of the conclusion $\int_{\mathbb{S}^2} v_i\nabla_{\mathbf{u}} v_i = 0$ and therefore $\int_{\mathbb{S}^2}\mathbf{v}\cdot(\nabla_{\mathbf{u}}\mathbf{v}) = 0$, which is the second part of the conclusion. □

Combine this lemma with (5.5) and (5.6), we prove the $L^2$ conservation (5.4).

To make generalization of the above procedure for $\|\mathbf{u}\|_{H^k}$, one takes spatial derivative $D^\alpha$ of (5.1) up to order $k$, i.e. $|\alpha| \leq k$, and inner-product it with $D^\alpha\mathbf{u}$.

$$\langle D^\alpha\mathbf{u}, D^\alpha\partial_t\mathbf{u}\rangle = -\langle D^\alpha\mathbf{u}, D^\alpha\nabla^\perp\Delta^{-1}\operatorname{curl}(\nabla_{\mathbf{u}}\mathbf{u})\rangle + \frac{1}{\varepsilon}\langle D^\alpha\mathbf{u}, D^\alpha\mathcal{L}[\mathbf{u}]\rangle. \tag{5.8}$$

The LHS equals $\frac{1}{2}\partial_t\|D^\alpha\mathbf{u}\|_{L^2}^2$. Then, one needs to make sure that every term on the RHS is bounded by the $H^k$ norm of $\mathbf{u}$. During this process, it is not trivial that the $1/\varepsilon$ term will vanish *completely* from the estimates, and therefore $T_\varepsilon$, the lifespan of classical $C^1$ solution may shrink to 0 as $\varepsilon \to 0$. A possible remedy is to use the skew-symmetric property of $\mathcal{L}$ as in (5.6) but we first need to show

$$\langle D^\alpha\mathbf{u}, D^\alpha\mathcal{L}[\mathbf{u}]\rangle = \langle D^\alpha\mathbf{u}, \mathcal{L}[D^\alpha\mathbf{u}]\rangle.$$

Unfortunately, this is not necessarily true because only a selective set of differential–integral operators on $\mathbb{S}^2$ commute with each other. To this end, we introduce the following lemma.

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**Lemma 5.2.** *Given integer $j \geq 0$. For sufficiently smooth vector field* $\mathbf{u}$ *on* $\mathbb{S}^2$ *with* $\operatorname{div} \mathbf{u} = 0$,

$$\langle \Delta^j \mathbf{u},\ \Delta^j \mathcal{L}[\mathbf{u}] \rangle = 0.$$
$$\langle \Delta^j \operatorname{curl} \mathbf{u},\ \Delta^j \operatorname{curl} \mathcal{L}[\mathbf{u}] \rangle = 0.$$

**Proof.** It is in fact sufficient to prove

$$\langle \Delta^k \mathbf{u}, \mathcal{L}[\mathbf{u}] \rangle = 0, \tag{5.9}$$

for any positive integer $k$, due to the symmetric property of $\Delta$, the commutativity of curl , $\Delta$, integrating-by-parts formula (A.20) and (2.2).

We first show that $\Delta$ and $\mathcal{L}$ share spherical harmonics as eigenfunctions and therefore they commute. By setting $\Psi = Y_l^m$ with $l \geq 1$ in (4.8), the spherical-harmonic expression of $\mathcal{L}$, we have

$$\mathcal{L}[\nabla^\perp Y_l^m] = \frac{-im}{l(l+1)} \nabla^\perp Y_l^m. \tag{5.10}$$

Then,

$$\begin{aligned} \Delta^k \mathcal{L}[\nabla^\perp Y_l^m] &= \Delta^k \frac{-im}{l(l+1)} \nabla^\perp Y_l^m \quad \ldots\text{by (5.10)} \\ &= \frac{-im}{l(l+1)} \nabla^\perp \Delta^k Y_l^m \quad \ldots\text{by commutativity (A.14)} \\ &= \mathcal{L}[\Delta^k \nabla^\perp Y_l^m] \quad \ldots\text{by (5.10)}. \end{aligned}$$

Therefore, for any incompressible flow $\mathbf{u} = \nabla^\perp \Psi = \nabla^\perp \sum_{l=1}^\infty \sum_{m=-l}^l \psi_l^m Y_l^m$, we have

$$\Delta^k \mathcal{L}[\mathbf{u}] = \mathcal{L}[\Delta^k \mathbf{u}]. \tag{5.11}$$

Combine this commutativity with the skew-symmetric property of $\mathcal{L}$ due to (5.7) and the symmetric property of $\Delta$ due to (A.22) to arrive at

$$\begin{aligned} \langle \Delta^k \mathbf{u}, \mathcal{L}[\mathbf{u}] \rangle &= -\langle \mathcal{L}[\Delta^k \mathbf{u}], \mathbf{u} \rangle = -\langle \Delta^k \mathcal{L}[\mathbf{u}], \mathbf{u} \rangle \\ &= -\langle \mathcal{L}[\mathbf{u}], \Delta^k \mathbf{u} \rangle, \end{aligned}$$

which leads to the conclusion (5.9). □

This lemma suggests that we replace $D^\alpha$ with $\Delta^j$ for $k = 2j$ (respectively, $\Delta^j$ curl for $k = 2j+1$) in (5.8). This is actually enough to meet the purpose of $H^k$ estimates, because, by the definition of $H^k$ norm in (4.7), we have

$$\begin{aligned} &\text{for } \operatorname{div} \mathbf{u} = 0, \qquad \|\mathbf{u}\|_{H^{2j}}^2 \approx \|\Delta^j \mathbf{u}\|_{L^2}^2 \\ &\text{and} \quad \|\mathbf{u}\|_{H^{2j+1}}^2 \approx \|\Delta^j \operatorname{curl} \mathbf{u}\|_{L^2}^2 \end{aligned} \tag{5.12}$$

where $A \approx B$ means $A/B$ is of uniform lower and upper bounds that only depend on $j$.

Now that the $1/\varepsilon$ term in (5.8) equals zero, we only need to estimate the nonlinear term, which contains derivatives up to order $k+1$, one higher order than we can handle. To close this gap, we apply Lemma 5.1 to obtain

$$\begin{aligned} &\langle D^\alpha \mathbf{u}, D^\alpha \nabla^\perp \Delta^{-1} \operatorname{curl}(\nabla_{\mathbf{u}} \mathbf{u}) \rangle \\ &\quad = \langle D^\alpha \mathbf{u}, (D^\alpha \nabla^\perp \Delta^{-1} \operatorname{curl}(\nabla_{\mathbf{u}} \mathbf{u}) - \nabla_{\mathbf{u}} D^\alpha \mathbf{u}) \rangle \end{aligned} \tag{5.13}$$

where $D^\alpha = \Delta^j \operatorname{curl}$ for $k = 2j+1$ (resp. $D^\alpha = \Delta^j$ for $k = 2j$).

Then, the commutator term on the RHS of (5.13) amounts to (with some help from (A.14))

$$\Delta^j \operatorname{curl}(\nabla_{\mathbf{u}} \mathbf{u}) - \nabla_{\mathbf{u}}(\Delta^j \operatorname{curl} \mathbf{u}) \quad \text{for } k = 2j+1$$
$$\left\{\text{resp. } \nabla^\perp \Delta^{j-1} \operatorname{curl}(\nabla_{\mathbf{u}} \mathbf{u}) - \nabla_{\mathbf{u}}(\Delta^j \mathbf{u}) \text{ for } k = 2j\right\}.$$

They contain derivatives up to order $k$ in the sense of the following lemma.

**Lemma 5.3.** *For integer* $k = 2j+1 \geq 3$ *(resp.* $k = 2j \geq 4$*) and incompressible vector field* $\mathbf{u}$ *with sufficient regularity,*

$$\|\Delta^j \operatorname{curl}(\nabla_{\mathbf{u}} \mathbf{u}) - \nabla_{\mathbf{u}}(\Delta^j \operatorname{curl} \mathbf{u})\|_{L^2} \leq C_k \|\mathbf{u}\|_{H_k}^2$$
$$\left\{\textit{resp. } \|\nabla^\perp \Delta^{j-1} \operatorname{curl}(\nabla_{\mathbf{u}} \mathbf{u}) - \nabla_{\mathbf{u}}(\Delta^j \mathbf{u})\|_{L^2} \leq C_k \|\mathbf{u}\|_{H_k}^2\right\}.$$

**Proof.** The key idea is that, although the highest derivatives in each term of the commutator are of $(k+1)$th order, they are canceled out by the subtraction in the commutator. We illustrate the details for $k = 2j+1$ with $j \geq 1$.

To make things easier, borrow the geometry of $\mathbb{R}^3$ and consider an open neighborhood of $\mathbb{S}^2$, called $\Omega$. For any scalar field $f$ defined on $\mathbb{S}^2$, we make a "normally constant" extension to $\Omega$

$$f(x, y, z) = f\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) \quad \text{with } r = \sqrt{x^2 + y^2 + z^2}.$$

Then, any normal derivative (i.e. derivative along the radial direction) of $f$ is zero. Make the same normally constant extension for vector field $\mathbf{v}$ too,

$$\mathbf{v}(x, y, z) = \mathbf{v}\left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right),$$

i.e., each Cartesian component of $\mathbf{v}$ is normally constant. Here, $\mathbf{v}$ is not necessarily tangent to $\mathbb{S}^2$.

Now, denote such normally constant extension by

$$f \overset{2\rightarrow 3}{=} f, \qquad \mathbf{v} \overset{2\rightarrow 3}{=} \mathbf{v}$$

without changing the names. The extension of differential operators, on the other hand, require notational changes. Let us use subscript 3 attached to an operator, e.g. $\Delta_3$, to indicate it is three dimensional. Then, one can use Calculus to show that, for normally constant $f$, $\mathbf{v}$,

$$\begin{aligned} &\Delta f \overset{2\rightarrow 3}{=} r^2 \Delta_3 f \\ &\nabla f \overset{2\rightarrow 3}{=} r \nabla_3 f \\ &\nabla_{\mathbf{u}} f \overset{2\rightarrow 3}{=} r \mathbf{u} \cdot \nabla_3 f \\ &\operatorname{div}\Big(\operatorname*{Proj}_{3\rightarrow 2} \mathbf{v}\Big) \overset{2\rightarrow 3}{=} r \operatorname{div}_3 \mathbf{v} - 2\mathbf{v} \cdot \mathbf{e}_r \\ &\operatorname{curl}\Big(\operatorname*{Proj}_{3\rightarrow 2} \mathbf{v}\Big) \overset{2\rightarrow 3}{=} r \operatorname{div}_3 (\mathbf{v} \times \mathbf{e}_r). \end{aligned}$$

Here, $\operatorname*{Proj}_{3\rightarrow 2} \mathbf{v}$ maps $\mathbf{v}$ from $\mathbb{R}^3$ to its tangent part on $\mathbb{S}^2$.

For the normally constant extension of covariant derivative $\nabla_{\mathbf{u}} \mathbf{u}$, we first observe that $r\mathbf{u} \cdot \nabla_3 \mathbf{u}$ is normally constant and thus $\nabla_{\mathbf{u}} \mathbf{u} \overset{2\rightarrow 3}{=} -(r\mathbf{u} \cdot \nabla_3 \mathbf{u}) \times \mathbf{e}_r \times \mathbf{e}_r$. Combine it with the extension of curl $\mathbf{u}$ from above and obtain

$$\operatorname{curl} \nabla_{\mathbf{u}} \mathbf{u} \overset{2\rightarrow 3}{=} r \operatorname{div}_3\big((r\mathbf{u} \cdot \nabla_3 \mathbf{u}) \times \mathbf{e}_r\big).$$

The above extension formulas allow us to work directly with Cartesian coordinates, in particular, the $\partial_x, \partial_y, \partial_z$ derivatives and their combinations. Going through lengthy but routine work of calculation using simple Calculus, one can obtain

$$\operatorname{curl}(\nabla_{\mathbf{u}} \mathbf{u}) - \nabla_{\mathbf{u}}(\operatorname{curl} \mathbf{u}) \overset{2\rightarrow 3}{=} \sum_{a,b=0}^{1} B(\nabla_3^a \mathbf{u}, \nabla_3^b \mathbf{u})$$

where $B(\cdot, \cdot)$ denotes some generic bilinear function with smooth coefficients. Also, $\nabla_3^a$ denotes $\partial_x, \partial_y, \partial_z$ derivatives and their combinations up to order $a$ and these derivatives are taken on the Cartesian components of $\mathbf{u}$. Notice that, although 2nd derivatives appear on the LHS, they are canceled out on the RHS.

Apply $\Delta^j$ on the previous equation and use the extension formulas to obtain

$$\Delta^j \operatorname{curl}(\nabla_{\mathbf{u}}\mathbf{u}) - \Delta^j\nabla_{\mathbf{u}}(\operatorname{curl}\mathbf{u}) \overset{2\to 3}{=} \sum_{\substack{a,b=0\\ a+b\le 2j+2}}^{2j+1} B(\nabla_3^a\mathbf{u},\ \nabla_3^b\mathbf{u}). \quad (5.14)$$

Observe again that $(2j+2) = (k+1)$th derivatives appear on the LHS but they are canceled out on the RHS. Also, notice that $a+b \le 2j+2$ implies $\min\{a,b\} \le j+1$. Thus, the $L^2(\mathbb{S}^2)$ norm of every term on the RHS is bounded by a constant times

$$\sum_{a=0}^{j+1}\sum_{b=0}^{2j+1} |\nabla_3^a\mathbf{u}|_{L^\infty(\mathbb{S}^2)}\|\nabla_3^b\mathbf{u}\|_{L^2(\mathbb{S}^2)}.$$

The expression $\nabla_3^a\mathbf{u}$ is not geometrically intrinsic to $\mathbb{S}^2$. However, due to our normally constant extension, the normal component of $\mathbf{u}$ and the normal derivatives of $\mathbf{u}$ are zero; so, we can replace all the $x, y, z$ derivatives with combinations of tangential derivatives on $\mathbb{S}^2$ and then find an upper bound for the above quantity

$$C|\mathbf{u}|_{W^{j+1,\infty}(\mathbb{S}^2)}\|\mathbf{u}\|_{H^{2j+1}(\mathbb{S}^2)}.$$

Due to Sobolev inequality (5.3), the $|\mathbf{u}|_{W^{j+1,\infty}(\mathbb{S}^2)}$ term of this quantity is bounded by the $\|\mathbf{u}\|_{H^{2j+1}(\mathbb{S}^2)}$ term, as long as $j \ge 1$. Thus, we establish $\|\mathbf{u}\|^2_{H^{2j+1}} = \|\mathbf{u}\|^2_{H^k}$ times a constant as an upper bound for the $L^2(\mathbb{S}^2)$ norm of (5.14)

$$\|\Delta^j \operatorname{curl}(\nabla_{\mathbf{u}}\mathbf{u}) - \Delta^j\nabla_{\mathbf{u}}(\operatorname{curl}\mathbf{u})\|_{L^2(\mathbb{S}^2)} \le C\|\mathbf{u}\|^2_{H^k}. \quad (5.15)$$

The same type of calculation works for estimating the $L^2$ norm of commutator $\Delta^j\nabla_{\mathbf{u}}(\operatorname{curl}\mathbf{u}) - \nabla_{\mathbf{u}}(\Delta^j\operatorname{curl}\mathbf{u})$. In particular, $(2j+2) = (k+1)$th derivatives are canceled out, so the same type of bound in (5.15) also works for this term

$$\|\Delta^j\nabla_{\mathbf{u}}(\operatorname{curl}\mathbf{u}) - \nabla_{\mathbf{u}}(\Delta^j\operatorname{curl}\mathbf{u})\|_{L^2(\mathbb{S}^2)} \le C\|\mathbf{u}\|^2_{H^k}.$$

Adding the above two estimates and applying the triangle inequality, we prove the first part of the conclusion. The second part for $k = 2j \ge 4$ can be proved without essential changes. □

We finish this section by stating and proving the following theorem for $H^k$ estimates of the solution independent of $\varepsilon$.

**Theorem 5.1.** *Consider the incompressible Euler equations (5.1), (5.2) on a rotating sphere $\mathbb{S}^2$ with div-free initial data $\mathbf{u}_0$. Given any integer $k \ge 3$, assume $\mathbf{u}_0 \in H^k(S^2)$. Then, there exists universal constants $C_0, T_0$ independent of $\varepsilon$ so that*

$$\|\mathbf{u}(t,\cdot)\|_{H^k} \le C_0\|\mathbf{u}_0\|_{H^k} \quad \textit{for any } t \in \left[0, \frac{T_0}{\|\mathbf{u}_0\|_{H^k}}\right].$$

**Proof.** The existence and uniqueness of $H^k$ solution is well established for general hyperbolic PDE systems that are symmetrizable (e.g. [25] and references therein). Assume the maximum lifespan of such solution is $T_\varepsilon$. Our goal is to show that $T_\varepsilon \ge T_0/\|\mathbf{u}_0\|_{H^k}$.

Set $D^\alpha = \Delta^j\operatorname{curl}$ for $k = 2j+1$ (resp. $D^\alpha = \Delta^j$ for $k = 2j$) in (5.8), apply Lemma 5.2 to make the $1/\varepsilon$ term vanish and apply (5.13) with Lemma 5.3 to estimate the tri-linear product,

$$\frac{1}{2}\partial_t\|D^\alpha\mathbf{u}\|^2_{L^2} \le C\|D^\alpha\mathbf{u}\|_{L^2}\|\mathbf{u}\|^2_{H^k}$$

where we also used the Cauchy–Schwartz inequality. By equivalence of $H^k$ norms as in (5.12), the above estimate leads to

$$\frac{1}{2}\partial_t\|D^\alpha\mathbf{u}\|^2_{L^2} \le C\|D^\alpha\mathbf{u}\|^3_{L^2}.$$

By the uniqueness of classical solutions, $\mathbf{u}_0 \equiv 0 \iff \mathbf{u}(t,\cdot) \equiv 0$; thus, we only deal with solution $\mathbf{u}$ with $\|D^\alpha\mathbf{u}\|_{L^2} \approx \|\mathbf{u}\|_{H^k} \ne 0$. Simplify the above estimate by dividing both sides with $\|D^\alpha\mathbf{u}\|^3_{L^2}$

$$\frac{\partial_t\|D^\alpha\mathbf{u}\|_{L^2}}{\|D^\alpha\mathbf{u}\|^2_{L^2}} \le C$$

$$\Longrightarrow -\partial_t\left(\|D^\alpha\mathbf{u}\|^{-1}_{L^2}\right) \le C$$

$$\Longrightarrow \|D^\alpha\mathbf{u}\|_{L^2} \le \frac{1}{\|D^\alpha\mathbf{u}_0\|_{L^2} - Ct}.$$

By the equivalence (5.12), one can manipulate the above estimate to find suitable values for $C_0, T_0$ as used the conclusion of the theorem. All generic constants in this proof are clearly independent of $\varepsilon$ and only depend on $k$. □

## 6. Proof of the main theorem

The Main Theorem 1.1 fits into the framework of Lemma 1.1 and we have established most of the ingredients in this framework. In particular, we have defined and proved properties of operator $\mathcal{L}$ in Definition 2.1, (4.8) and Lemma 4.1; and defined operator $\prod_{\mathrm{null}\{\mathcal{L}\}}$ in Lemma 3.2. We have proved the key estimate (1.7) in Theorem 4.1 and $\varepsilon$-independent $H^k$ estimates for the solution in Theorem 5.1. In this section, we complete the proof of Theorem 1.1 by verifying regularity assumptions (1.5) and (1.6), and then establishing estimates for $M_0$ in (1.3).

First, by fitting (5.1) into the framework of Lemma 1.1, we set $\mathcal{L}$ as defined in (5.2) and

$$f := \nabla^\perp\Delta^{-1}\operatorname{curl}(\nabla_{\mathbf{u}}\mathbf{u}).$$

Given initial data $\mathbf{u}_0 \in H^k(\mathbb{S}^2)$, by Theorem 5.1, we have

$$\|\mathbf{u}(t,\cdot)\|_{H^k} \le C_0\|\mathbf{u}_0\|_{H^k} \quad \text{for any } t \in \left[0, \frac{T_0}{\|\mathbf{u}_0\|_{H^k}}\right]. \quad (6.1)$$

Then, by the normally constant extension introduced in the proof of Lemma 5.3, one can show that

$$\|f\|_{H^{k-1}} \le C\|\mathbf{u}\|^2_{H^k} \quad \text{and} \quad \|\mathcal{L}[\mathbf{u}]\|_{H^{k+1}} \le C\|\mathbf{u}\|_{H^k}. \quad (6.2)$$

Set Hilbert spaces

$$X_1 := H^{k-3}(\mathbb{S}^2), \qquad X_2 := H^{k-1}(\mathbb{S}^2),$$

so that (1.7) holds true according to Theorem 4.1.

The assumption (1.5) is verified by (6.1), (6.2). Note that the time-continuity part of (1.5) is due to the calculation,

$$\left|\|\mathbf{u}(t+\tau,\cdot) - \mathbf{u}(t,\cdot)\|_{H^k}\right| \le \left|\|\mathbf{u}(t+\tau,\cdot)\|_{H^k} - \|\mathbf{u}(t,\cdot)\|_{H^k}\right| = \left|\int_t^{t+\tau}\partial_t\|\mathbf{u}\|_{H^k}\right|$$

and the $\partial_t\|\mathbf{u}\|_{H^k}$ term on the RHS can be shown to be uniformly bounded following the proof of Theorem 5.1. Then, one lets $\tau \to 0$ to show the time-continuity of $\mathbf{u}$. Likewise, this can be done for $\mathcal{L}[\mathbf{u}]$ and $f$.

As for the commutativity of $\mathcal{L}$ and $\int dt$ in (1.6), we use the time-continuity of $\mathbf{u}$ and $\mathcal{L}[\mathbf{u}]$ in terms of $H^k$ norm to rewrite the time integrals as limits of Riemann sums

$$\int_0^T \mathcal{L}[\mathbf{u}]\,dt = \lim_{N\to 0}\sum_i \mathcal{L}[\mathbf{u}(t_i,\cdot)]\delta t$$

$$\mathcal{L}\left[\int_0^T \mathbf{u}\,dt\right] = \mathcal{L}\left[\lim_{N\to 0}\sum_i \mathbf{u}(t_i,\cdot)\delta t\right]$$

where $\delta t = T/N$ and $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$ form an equi-partition of $[0, T]$. In the RHS of the second equality, $\mathcal{L}$ and lim commute because $\mathcal{L}$ is a continuous mapping on $H^k$. Thus, the LHS of the two above equalities are equal.

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Finally, by estimates (6.1), (6.2), the constant $M$ in Lemma 1.1 is bounded by $\|\mathbf{u}\|_{H^k} \le C_0\|\mathbf{u}_0\|_{H^k}$ and the constant $M'$ is bounded by $\|f\|_{H^{k-1}} \le C\|\mathbf{u}\|^2_{H^k} \le C'\|\mathbf{u}_0\|^2_{H^k}$. These two bounds validate the use of constant $M_0$ in (1.3).

The proof of Theorem 1.1 is complete.

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## Appendix. Preparation in differential geometry

Let $\mathbb{M}$ denote a 2-dimensional, compact, Riemannian manifold without boundary, typically the unit sphere $\mathbb{M} = \mathbb{S}^{3-1}$, endowed with metric $\mathfrak{g}$ inherited from the embedding Euclidean space $\mathbb{R}^3$. Let $p \in \mathbb{M}$ denote a point with local coordinates $(p_1, p_2)$.

Any vector field $\mathbf{u}$ in the tangent bundle $T\mathbb{M}$ is identified with a field of *directional differentials*, which is written in local coordinates as

$$\mathbf{u} = \sum_i a^i \frac{\partial}{\partial p_i}.$$

We use the notation

$$\nabla_{\mathbf{u}} f := \sum_i a^i \frac{\partial f}{\partial p_i} \tag{A.1}$$

to denote the directional derivative of a scalar-valued function $f$ in the direction of $\mathbf{u}$. Using the orthogonal projection $\text{Proj}_{T\mathbb{R}^3\to T\mathbb{M}}$ induced by the Euclidean metric of $\mathbb{R}^3$, we define the covariant derivative of a vector field $\mathbf{v} \in T\mathbb{M}$ along another vector field $\mathbf{u} \in T\mathbb{M}$,

$$\nabla_{\mathbf{u}}\mathbf{v} := \operatorname*{Proj}_{T\mathbb{R}^3\to T\mathbb{M}} \sum_{i=1}^{3} (\nabla_{\mathbf{u}} v_i)\mathbf{e}_i. \tag{A.2}$$

Here, $\mathbf{v}$ is expressed in an orthonormal basis of $\mathbb{R}^3$ as $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$.

The metric $\mathfrak{g}$ is identified with a $(0, 2)$ tensor, simply put, an $2 \times 2$ matrix $(g_{ij})_{2\times 2}$ in local coordinates. Thus, the vector inner product follows

$$\mathfrak{g}\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}\right) = g_{ij} \quad \text{for } 1 \le i, j \le 2.$$

The definitions and basic properties of *differential forms* can be found in e.g. [21]. We only sketch the following facts that will be used in this section. A differential $k$-form $\beta$, at given point $p \in \mathbb{M}$, maps any $k$-tuple of tangent vectors to a scalar. In particular, a 0-form is identified with a scalar-valued function. The 1-form $dp_i$ in local coordinates satisfies $dp_i(\frac{\partial}{\partial p_j}) = \delta_{ij}$. The *exterior differential* $d$ maps a $k$-form to a $(k+1)$ form. For example, for 0-form $f$, $df = \sum_i \frac{\partial f}{\partial p_j} dp_i$ so that $df(\mathbf{u}) = \nabla_{\mathbf{u}} f$. The *wedge product* of a $k$-form $\alpha$ and $l$-form $\beta$, denoted by $\alpha \wedge \beta$, is a $(k+l)$ form. It is skew-commutative in the sense that $\alpha \wedge \beta = (-1)^{kl}\beta \wedge \alpha$.

### A.1. Hodge theory [21,22]

The Hodge $^*$-operator, defined in an orthonormal basis[4] $\frac{\partial}{\partial p_1}$, $\frac{\partial}{\partial p_2}$ in a sub-region of $\mathbb{M}$, satisfies

$$*dp_1 = dp_2, \qquad *dp_2 = -dp_1, \qquad *1 = dp_1 \wedge dp_2,$$
$$*(dp_1 \wedge dp_2) = 1.$$

[4] The existence of such basis is guaranteed by the Gram–Schmidt orthogonalization process.

It is easy to see that Hodge $^*$-operator maps between $k$-forms and $(n-k)$ form. And its square, $**$ amounts to identity or $(-1)$ times identity.

Using the Hodge star operator, we define the co-differential for any $k$-forms $\alpha$ in an $n$-dimensional manifold,

$$\textbf{codifferential}: \ \delta\alpha := (-1)^k *^{-1} d * \alpha = (-1)^{n(k+1)+1} * d * \alpha,$$

and in particular, for $n = 2$,

$$\delta\alpha = - * d * \alpha.$$

So, $\delta$ maps a $k$-form to a $(k-1)$ form.

The Hodge Laplacian (a.k.a. Laplace–Beltrami operator and Laplace-de Rham operator) is then defined by

$$\Delta_H := d\delta + \delta d. \tag{A.3}$$

In particular, for a scalar-valued function $f$ in a local basis $\left\{\frac{\partial}{\partial p_i}\right\}$ with metric $\mathfrak{g}$, it is identified as

$$\Delta_H f = -\frac{1}{\sqrt{|\mathfrak{g}|}} \sum_{i,j} \partial_i(\sqrt{|\mathfrak{g}|} g^{ij} \partial_j f)$$

where $(g^{ij})$ is the matrix inverse of $(g_{ij})$. Thus, on a surface $\mathbb{M}$, the Hodge Laplacian $\Delta_H$ defined in (A.3) amounts to the surface Laplacian $\Delta_{\mathbb{M}}$ times $(-1)$. In particular, if $\mathbb{M}$ is a two-dimensional surface, then

$$\text{for scalar function } f, \quad \Delta_{\mathbb{M}} f = -\delta df = *d * df \tag{A.4}$$

since $\delta f = 0$ for a 0-form $f$. For consistency, we also fix the surface Laplacian $\Delta_{\mathbb{M}}$ of 1-forms as the Hodge Laplacian $\Delta_H$ times $(-1)$,

$$\begin{aligned} \text{for 1-form } \alpha, \quad \Delta_{\mathbb{M}}\alpha &= -(d\delta + \delta d)\alpha \\ &= (d * d * + * d * d)\alpha. \end{aligned} \tag{A.5}$$

For now on, we will use $\Delta$ for $\Delta_{\mathbb{M}}$.

The *Hodge decomposition* theorem in its most general form states that for any $k$-form $\omega$ on an oriented compact Riemannian manifold, there exist a $(k-1)$-form $\alpha$, $(k+1)$-form $\beta$ and a harmonic $k$-form $\gamma$ satisfying $\Delta_H\gamma = 0$, s.t.

$$\omega = d\alpha + \delta\beta + \gamma.$$

In particular, for any 1-form $\omega$ on a 2-dimensional manifold with the 1st Betti number 0 (loosely speaking, there is no "holes"), there exist two scalar-valued functions $\Phi$, $\Psi$ such that

$$\omega = d\Phi + \delta(*\Psi) = d\Phi - *d\Psi. \tag{A.6}$$

Here, the third term drops out of the RHS of (A.6) because, by the Hodge theory, the dimension of the space of harmonic $k$-forms on $M$ equals the $k$-th Betti number of $M$. In the cohomology class containing the unit sphere $\mathbb{S}^2$, the 0th, 1st and 2nd Betti numbers are respectively 1, 0, 1. In other words, the only harmonic 1-form on $\mathbb{S}^2$ is zero.

### A.2. In connection with vector fields

In a Riemannian manifold, there is a 1-to-1 correspondence, induced by the metric $\mathfrak{g}$, between vectors and 1-forms. They are the so called "musical isomorphisms" demoted by $\flat$ and $\sharp$. For any vector fields $\mathbf{u}$, $\mathbf{v}$, the 1-form $\mathbf{u}^\flat$ satisfies,

$$\mathbf{u}^\flat(\mathbf{v}) = \mathfrak{g}(\mathbf{u}, \mathbf{v}), \qquad (\mathbf{u}^\flat)^\sharp = \mathbf{u}.$$

In a (local) orthonormal basis, $\flat$ and $\sharp$ map between vectors and 1-forms with identical coordinates.

In a 2-dimensional Riemannian manifold, in order to define the divergence and curl of a vector field $\mathbf{u} \in T\mathbb{M}$, we use $\flat$ to map it to 1-form and then apply $\delta$ and $d$ to obtain the scalar fields,[5]

$$\textbf{divergence},\ \operatorname{div}\mathbf{u} := -\delta(\mathbf{u}^\flat) = *d*(\mathbf{u}^\flat) \tag{A.7}$$

$$\textbf{curl},\ \operatorname{curl}\mathbf{u} := -\delta(*\mathbf{u}^\flat) = -*d(\mathbf{u}^\flat). \tag{A.8}$$

For a scalar field $f$, we define gradient and its $\pi/2$ rotation as

$$\textbf{gradient},\ \nabla f := (df)^\sharp \tag{A.9}$$

$$\textbf{rotated gradient},\ \nabla^\perp f := -(*df)^\sharp. \tag{A.10}$$

We also define the counterclockwise $\pi/2$ rotation operator $^\perp$ acting on a vector field as

$$\mathbf{u}^\perp := -(*\mathbf{u}^\flat)^\sharp \tag{A.11}$$

so that, consistently, $\nabla^\perp f = (\nabla f)^\perp$ and $\operatorname{div}\mathbf{u} = \operatorname{curl}\mathbf{u}^\perp$.

Combine these definitions with that of the surface Laplacian (A.4), (A.5) to obtain

$$\textbf{Laplacian of scalar},\ \Delta f := \operatorname{div}\nabla f = \operatorname{curl}\nabla^\perp f \tag{A.12}$$

$$\textbf{Laplacian of vector},\ \Delta\mathbf{u} := \nabla\operatorname{div}\mathbf{u} + \nabla^\perp\operatorname{curl}\mathbf{u}. \tag{A.13}$$

An immediate consequence is that

$$\Delta \text{ commutes with each one of } \operatorname{div},\ \operatorname{curl},\ \nabla,\ \nabla^\perp. \tag{A.14}$$

To this end, the vector-field version of Hodge decomposition (A.6) becomes, for smooth vector field $\mathbf{u}$ on $\mathbb{S}^2$, there exist scalar fields $\Phi$, $\Psi$, s.t.

$$\textbf{Hodge decomposition},\ \mathbf{u} = \nabla\Phi + \nabla^\perp\Psi. \tag{A.15}$$

We note that, by the virtue of (A.12), the decomposition satisfies

$$\operatorname{div}\mathbf{u} = \Delta\Phi, \qquad \operatorname{curl}\mathbf{u} = \Delta\Psi.$$

It is also easy to use these definitions to verify the following properties,

$$\operatorname{curl}\nabla f = \operatorname{div}\nabla^\perp f = 0 \tag{A.16}$$

due to $dd = 0$ and $\delta\delta = 0$; and,

$$\operatorname{curl}(f\mathbf{u}^\perp) = \operatorname{div}(f\mathbf{u}) = \nabla f\cdot\mathbf{u} + f\operatorname{div}\mathbf{u}, \tag{A.17}$$

as a consequence of the product rule for differential $d$.

### A.3. In connection with surface integrals

The integral of scalar field $f$ over an $n$-dimensional, oriented Riemannian manifold $\mathbb{M}$, defined in differential-geometric terms (e.g. [21, Section 4.10]), is the integral of the $n$-form $*f$

$$\int_{\mathbb{M}} f = \int_{\mathbb{M}} *f. \tag{A.18}$$

Note that any $n$-form defines a measure on $\mathbb{M}$.

For the most general case, the inner product of $k$-forms $\alpha_1, \alpha_2$ is defined as

$$\langle\alpha_1, \alpha_2\rangle := \int_{\mathbb{M}} (\alpha_1 \wedge *\alpha_2).$$

In particular, for vector fields $\mathbf{u}$, $\mathbf{v}$, we map them to 1-forms and define $L^2(\mathbb{M})$ inner product as

$$\langle\mathbf{u}, \mathbf{v}\rangle_{L^2(\mathbb{M})} := \langle\mathbf{u}^\flat, \mathbf{v}^\flat\rangle.$$

This coincides exactly with the more conventional definition

$$\langle\mathbf{u}, \mathbf{v}\rangle_{L^2(\mathbb{M})} := \int_{\mathbb{M}} \mathfrak{g}(\mathbf{u}, \mathbf{v})$$

where $\mathfrak{g}$ is the given metric—in Euclidean geometry, $\mathfrak{g}(\mathbf{u}, \mathbf{v}) = \mathbf{u}\cdot\mathbf{v}$.

It then follows from the celebrated Stokes theorem (e.g. [21]) that, in the case when $\mathbb{M}$ has no boundary, the codifferential is the adjoint of exterior differential w.r.t. $L^2(\mathbb{M})$ inner product

$$\langle d\alpha, \beta\rangle = \langle\alpha, \delta\beta\rangle.$$

This duality relation, together with definitions (A.7)–(A.13), immediately leads to the following "integrating-by-parts" formulas on a surface $\mathbb{M}$, which generalize the Green's identities and give adjoint relations of the differential operators we just defined.

$$\langle\nabla f, \mathbf{u}\rangle = -\langle f, \operatorname{div}\mathbf{u}\rangle, \tag{A.19}$$

$$\langle\nabla^\perp f, \mathbf{u}\rangle = -\langle f, \operatorname{curl}\mathbf{u}\rangle, \tag{A.20}$$

$$\langle f, \Delta h\rangle = \langle\Delta f, h\rangle = -\langle\nabla f, \nabla h\rangle, \tag{A.21}$$

$$\langle\mathbf{u}, \Delta\mathbf{v}\rangle = \langle\Delta\mathbf{u}, \mathbf{v}\rangle = -\langle\operatorname{div}\mathbf{u}, \operatorname{div}\mathbf{v}\rangle - \langle\operatorname{curl}\mathbf{u}, \operatorname{curl}\mathbf{v}\rangle. \tag{A.22}$$

Here, for simplicity, the $L^2(\mathbb{M})$ subscript is omitted from all $\langle,\rangle$ expressions. The $\langle,\rangle$ of scalar fields is also easily understood.

### A.4. Local expression in terms of spherical coordinates for $\mathbb{M} = \mathbb{S}^2$

Although the proofs in this article are independent of any local coordinate systems, we provide here, for interested readers, the spherical-coordinate forms of some of the differential operators defined above.

Let $\phi$ denote the longitude and $\theta$ the colatitude of a point on a sphere. Let $\mathbf{e}_\phi$, $\mathbf{e}_\theta$ denote the unit tangent vectors in the increasing directions of $\phi$ and $\theta$. Then, at point $p$ away from the poles,

$$\partial_\phi = \sin\theta\,\mathbf{e}_\phi, \qquad \partial_\theta = \mathbf{e}_\theta,$$

namely,

$$\frac{1}{\sin\theta}\partial_\phi \quad\text{and}\quad \partial_\theta \text{ form an orthonormal basis of } T\mathbb{M}_p.$$

Therefore, the musical isomorphisms, in $\phi$, $\theta$ coordinates, satisfy

$$\left(\frac{1}{\sin\theta}\partial_\phi\right)^\flat = \sin\theta\, d\phi \quad\text{and}$$

$$(\partial_\theta)^\flat = d\theta \text{ form an orthonormal basis of } T^*\mathbb{M}_p.$$

In this context, the Hodge $*$-operator satisfies, with smooth scalar fields $f_1, f_2, f$,

$$\text{for 1-forms, } *(f_1 d\phi + f_2 d\theta) = \frac{f_1}{\sin\theta}d\theta - f_2\sin\theta\, d\phi$$

$$\text{for 0-forms and 2-forms, } *(f d\phi\wedge d\theta) = \frac{f}{\sin\theta},$$

$$*f = f\sin\theta\, d\phi\wedge d\theta.$$

Combining the last equation with (A.18), we have the spherical expression for the integral of scalar field $f$ over $\mathbb{S}^2$,

$$\int_{\mathbb{S}^2} f = \int_0^\pi\int_0^{2\pi} f(\theta,\phi)\sin\theta\, d\phi d\theta,$$

and inner product of vector fields, $\mathbf{u} = u_1\mathbf{e}_\phi + u_2\mathbf{e}_\theta$, $\mathbf{v} = v_1\mathbf{e}_\phi + v_2\mathbf{e}_\theta$,

$$\langle\mathbf{u}, \mathbf{v}\rangle_{L^2(\mathbb{S}^2)} = \int_0^\pi\int_0^{2\pi} (u_1v_1 + u_2v_2)\sin\theta\, d\phi d\theta.$$

[5] Since 0-forms are identified with scalar fields, we use Hodge *-operator to map between 0-forms and 2-forms.

The differential operators defined in (A.7)–(A.13) then become, for scalar field $f$

$$\nabla f = \frac{1}{\sin\theta}\partial_\phi f\,\mathbf{e}_\phi + \partial_\theta f\,\mathbf{e}_\theta$$

$$\nabla^\perp f = \partial_\theta f\,\mathbf{e}_\phi - \frac{1}{\sin\theta}\partial_\phi f\,\mathbf{e}_\theta$$

$$\Delta f = \frac{1}{\sin^2\theta}(\partial_\phi^2 f + \sin\theta\,\partial_\theta(\sin\theta\,\partial_\theta f)),$$

and

for vector field $\mathbf{u} = u_1\mathbf{e}_\phi + u_2\mathbf{e}_\theta$

$$\operatorname{div}\mathbf{u} = \frac{1}{\sin\theta}(\partial_\phi u_1 + \partial_\theta(u_2\sin\theta))$$

$$\operatorname{curl}\mathbf{u} = \frac{1}{\sin\theta}(\partial_\phi u_2 - \partial_\theta(u_1\sin\theta)).$$

The surface Laplacian of $\mathbf{u}$ can also be expressed using (A.13) and the formulas above.

The directional derivative of a scalar, (A.1), can be expressed as

$$\nabla_{\mathbf{u}} f = \frac{1}{\sin\theta}u_1\partial_\phi f + u_2\partial_\theta f$$

and the covariant derivative $\nabla_{\mathbf{u}}\mathbf{v}$ can be expressed accordingly using the above formula and (A.2).

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