

TIME-AVERAGES OF FAST OSCILLATORY SYSTEMS

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ABSTRACT. Time-averages are common observables in analysis of experimental data and numerical simulations of physical systems. We describe a straightforward framework for studying time-averages of dynamical systems whose solutions exhibit fast oscillatory behaviors. Time integration averages out the oscillatory part of the solution that is caused by the large skew-symmetric operator. Then, the time-average of the solution stays close to the kernel of this operator. The key assumption in this framework is that the inverse of the large operator is a bounded mapping between certain Hilbert spaces modular the kernel of the operator itself. This assumption is verified for several examples of time-dependent PDEs.

1. Introduction. Time-averaging is a common method to process and analyze observational data and numerical simulations alike. Patterns arise after time-averaging, which in turn affects long term dynamics of physical systems. For example, in the geophysical literature, zonal flow emerges in time-averages of data and/or simulations over time intervals from years to decades long ([9, 11, 13]). This pattern can be directly observed on Jupiter ([14]).

In this paper, we study such phenomena in the following mathematical framework. Consider a dynamical system

$$U_t = \frac{1}{\varepsilon} \mathcal{L}[U] + F, \quad (1)$$

where $0 < \varepsilon \ll 1$ is a scaling constant, \mathcal{L} is a linear operator independent of time and F includes nonlinear and source term. When \mathcal{L} is skew-symmetric, the system is nondissipative and exhibits highly oscillatory behavior “outside” the kernel of \mathcal{L} . We quantify such behavior in terms of time-averages of U . Several key hypotheses are outlined below for time-averaging to work; and we will verify these hypotheses for three time-dependent PDE systems of multiscale nature.

Let us assume a priori that, for some Hilbert spaces X_1, X_2 ,

$$U \in C([0, T], X_1 \cap X_2), \quad \mathcal{L}[U] \in C([0, T], X_2), \quad F \in C([0, T], X_2).$$

Let operator $\Pi_{\ker\{\mathcal{L}\}} : X_1 \rightarrow X_1$ denote (some) projection onto the kernel of \mathcal{L} . Then, under the hypotheses

$$\|U - \Pi_{\ker\{\mathcal{L}\}} U\|_{X_1} \leq C \|\mathcal{L}[U]\|_{X_2}, \quad (2)$$

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and

$$\int_0^T \mathcal{L}[U] dt = \mathcal{L}\left[\int_0^T U dt\right], \quad (3)$$

holds true the following estimate on the time-average of U ,

$$\left\| \frac{1}{T} \int_0^T U dt - \frac{1}{T} \int_0^T \Pi_{\ker\{\mathcal{L}\}} U dt \right\|_{X_1} \leq \varepsilon C \left(\frac{2M}{T} + M' \right) \quad (4)$$

where constants $M := \max_{t \in [0, T]} \|U(t)\|_{X_2}$ and $M' := \max_{t \in [0, T]} \|F(t)\|_{X_2}$. Note that the skew-symmetry of \mathcal{L} implies M, M' can be chosen to be bounded uniformly in ε on time interval $[0, T]$ which is also independent of ε .

The idea is straightforward. We simply transform the original system (1) into

$$U_t = \frac{1}{\varepsilon} \mathcal{L}[U - \Pi_{\ker\{\mathcal{L}\}} U] + F$$

and integrate both sides in time

$$U(T) - U(0) = \frac{1}{\varepsilon} \int_0^T \mathcal{L}[U - \Pi_{\ker\{\mathcal{L}\}} U] dt + \int_0^T F dt$$

Then, we apply (3) to rewrite the RHS,

$$U(T) - U(0) = \frac{1}{\varepsilon} \mathcal{L}\left[\int_0^T U dt - \int_0^T \Pi_{\ker\{\mathcal{L}\}} U dt\right] + \int_0^T F dt.$$

Separate the term with $1/\varepsilon$ from the rest and multiply both sides by ε ,

$$\mathcal{L}\left[\int_0^T U dt - \int_0^T \Pi_{\ker\{\mathcal{L}\}} U dt\right] = \varepsilon \left(U(T) - U(0) - \int_0^T F dt \right).$$

Measure both sides in X_2 norm and estimate the RHS using M and M' ,

$$\left\| \mathcal{L}\left[\int_0^T U dt - \int_0^T \Pi_{\ker\{\mathcal{L}\}} U dt\right] \right\|_{X_2} \leq \varepsilon (2M + M'T). \quad (5)$$

Finally, apply estimate (2) on the left side

$$\left\| \int_0^T U dt - \int_0^T \Pi_{\ker\{\mathcal{L}\}} U dt \right\|_{X_1} \leq C \left\| \mathcal{L}\left[\int_0^T U dt - \int_0^T \Pi_{\ker\{\mathcal{L}\}} U dt\right] \right\|_{X_2}$$

and combine it with (5) to arrive at (4).

Remark 1. The key hypothesis (2) is automatically true in a finite-dimensional space, say $X_1 = X_2 = \mathbb{R}^n$. In such case, (2) amounts to the boundedness of $\mathcal{L}^{-1} : \text{image}\{\mathcal{L}\} \mapsto \mathbb{R}^n / \ker\{\mathcal{L}\}$.

Remark 2. In a much more general context, we can loosen the a priori assumptions in (3) and Remark 3. The time integrals $\int_0^T U dt$ and $\int_0^T \mathcal{L}[U] dt$ can be defined as Bochner integrals ([1, pp. 6]) so that a sufficient condition for the commutative hypothesis (3) is that \mathcal{L} is a closed operator.

Remark 3. We focus on estimates rather than solution regularity in this paper. In particular, all Sobolev norms H^k are sufficiently smooth with suitable values of k and we always assume $U \in C([0, T], X_1 \cap X_2)$, $\mathcal{L}[U] \in C([0, T], X_2)$ and $F \in C([0, T], X_2)$.

When (1) governs the dynamics of a velocity field u , estimate (4) offers a simple explanation why the transporting properties of u are dictated by the projected component $\Pi_{\ker\{\mathcal{L}\}} u$. A typical example in such context is the incompressible approximation of compressible fluids.

We now illustrate the idea using a scalar transport equation

$$\partial_t s + u \cdot \nabla s = 0, \quad s_0(\cdot) \in H^k(\Omega) \quad (6)$$

for k sufficiently large. For simplicity, we assume Ω to be whole space \mathbb{R}^N or torus \mathbb{T}^N so that boundary condition is not a concern (for cases with boundary conditions, cf. [5, Corollary 1.4]). Define $\bar{u} := \Pi_{\ker\{\mathcal{L}\}} u$ and replace u with \bar{u} in (6) to obtain an approximate transport equation

$$\partial_t \bar{s} + \bar{u} \cdot \nabla \bar{s} = 0, \quad \bar{s}_0(\cdot) = s_0(\cdot). \quad (7)$$

Subtract (7) from (6), and rewrite in terms of $\delta := s - \bar{s}$

$$\partial_t \delta + \bar{u} \cdot \nabla \delta + (u - \bar{u}) \cdot \nabla s = 0. \quad (8)$$

Then, introduce time-average

$$\xi(t, \cdot) := \int_0^t (u - \bar{u}) \cdot \nabla s \, d\tau, \quad (9)$$

rewrite (8) as $\partial_t \delta + \bar{u} \cdot \nabla \delta + \partial_t \xi = 0$ and further

$$\partial_t (\delta + \xi) + \bar{u} \cdot \nabla (\delta + \xi) - \bar{u} \cdot \nabla \xi = 0, \quad \delta_0(\cdot) = 0.$$

Note that this is a transport equation for $(\delta + \xi)$ and thus energy method can be adopted to estimate the size of $(\delta + \xi)$. If, in addition, ξ is of order $O(\varepsilon)$ in certain spatial norms, then one can show $\delta = s - \bar{s}$ is also of order $O(\varepsilon)$ and therefore justify (7) as an approximation of (6).

For simplicity, we *formally* argue that ξ is indeed of order $O(\varepsilon)$, provided *some* spatial norms $\|u\|, \|\bar{u}\|, \|s\|, \|\bar{s}\|$ are a priori $O(1)$. In fact, perform integrating by parts with respect to time on (9)

$$\xi(t, \cdot) = w \cdot \nabla s \Big|_0^t - \int_0^t w \cdot \nabla (\partial_t s) \, d\tau \quad \text{where} \quad w(t, \cdot) := \int_0^t (u - \bar{u}) \, d\tau. \quad (10)$$

Estimate w using (4) and obtain, formally,

$$\|w\| \sim O(\varepsilon).$$

Then, use (6) to replace $\partial_t s$ in (10) with spatial derivatives and arrive at, formally,

$$\|\xi\| \sim O(\varepsilon).$$

A detailed application of the above method can be found in e.g. [5].

The rest of this article is dedicated to three PDE systems of multiscale nature, all of which are dealt with in the framework of (1) — (4). Thanks to the abundant results on solution regularities for these PDEs, we will always assume energy estimates are available a priori. The focus is therefore on $\mathcal{L}, \ker\{\mathcal{L}\}, \Pi_{\ker\{\mathcal{L}\}}$ and establishing the key estimate (2). The first two systems model geophysical flows on a fast rotating sphere: one is incompressible Euler equations and the other shallow water equations. Our argument will reveal that the kernels $\ker\{\mathcal{L}\}$ for these two systems are intimately related; and there is an interesting relation between the respective estimates (2) for these systems. In particular, we rigorously prove and quantify that the zonal flow pattern, as discussed above, arises from performing time-averaging

on geophysical flows in a *spherical* domain. The third PDE system is compressible Euler equations in a bounded domain. Our argument confirms the incompressible approximation of compressible flows, even in the case with ill-prepared initial data and nondissipative boundary conditions.

2. Euler Equations on a Fast Rotating Sphere. Consider the unit sphere \mathbb{S}^2 as the spatial domain. The incompressible Euler equations under Coriolis force are

$$\partial_t u + \nabla_u u + \nabla P = \frac{z}{\varepsilon} u^\perp, \quad \operatorname{div} u = 0 \quad (11)$$

where constant $\varepsilon \ll 1$, called the Rossby number, scales like the frequency of the frame's rotation and Cartesian coordinate z indicates how the Coriolis parameter varies along the meridional direction. Due to the \mathbb{S}^2 domain, we only use geometrically intrinsic notations: convection term $\nabla_u u$ is understood as the covariant derivative¹; rotational term u^\perp denotes the counterclockwise $\pi/2$ -rotation of u on \mathbb{S}^2 ; and, ∇^\perp will be used below for the counterclockwise $\pi/2$ -rotation of gradient ∇ on \mathbb{S}^2 .

We let θ denote the colatitude and ϕ the longitude. Also, let \mathbf{e}_ϕ be the unit vector in the zonal direction of increasing longitude and \mathbf{e}_θ be the unit vector in the meridional direction of increasing colatitude.

In [6, Theorem 1.1], the authors prove the following theorem.

Theorem 2.1. *Consider the incompressible Euler equations (11) with solution in $C([0, T]; H^k(\mathbb{S}^2))$ for $k \geq 3$. Then, there exist a function $f(\cdot) : [-1, 1] \mapsto \mathbb{R}$ and a universal constant C , only depending on k , s.t.*

$$\left\| \frac{1}{T} \int_0^T u \, dt - \nabla^\perp f(z) \right\|_{H^{k-3}(\mathbb{S}^2)} \leq C\varepsilon \left(\frac{2M}{T} + M^2 \right).$$

Here, $M := \max_{t \in [0, T]} \|u(t, \cdot)\|_{H^k}$. In spherical coordinates, the approximation $\nabla^\perp f(z)$ is

$$\nabla^\perp f(z) = -f'(\cos \theta) \sin \theta \mathbf{e}_\phi,$$

which is a **longitude-independent zonal flow**.

Simulations and observations have persistently shown that coherent anisotropy favoring *zonal flows* appears ubiquitously in planet scale circulations. For a partial list of computational results, we mention [9] for 3D models, [22, 17, 12, 21, 10] for 2D models, and references therein. These highly resolved, eddy-permitting simulations are made possible by rapid developments of high performance computing. On the other hand, we have observed zonal flow patterns (bands and jets) on giant planets for hundreds of years, which has attracted considerable interests recently thanks to spacecraft missions and the launch of the Hubble Space Telescope (e.g. [11], [18]). There are also observational data in the oceans on Earth showing persistent zonal flow patterns (e.g. [19, 20, 13]).

We now recap the proof for this theorem, in the abstract framework of (1) — (4). The main task is to identify \mathcal{L} , $\ker\{\mathcal{L}\}$, $\Pi_{\ker\{\mathcal{L}\}}$ and to establish the key estimate (2). Note that the commutability condition (3) is true since \mathcal{L} as defined in (12) below is a bounded and of course closed operator in any H^k space — consult Remark 2.

¹To define $\nabla_u v$ for smooth vector fields $u, v \in T\mathbb{S}^2$, we extend both of them smoothly to \mathbb{R}^3 , calculate $u \cdot \nabla v$ in Cartesian coordinates and project $u \cdot \nabla v$ back onto the tangent bundle $T\mathbb{S}^2$.

In [6], the authors employ differential geometric tools to reformulate (11) as,

$$\partial_t u + \nabla^\perp \Delta^{-1} \operatorname{curl}(\nabla_u u) = \frac{1}{\varepsilon} \mathcal{L}[u],$$

where

$$\mathcal{L}[u] := \nabla^\perp \Delta^{-1} \operatorname{curl}(zu^\perp) = \nabla^\perp \Delta^{-1}(u \cdot \nabla z),$$

for div-free velocity field u on \mathbb{S}^2 . Then, it is easy to see that

$$\text{For div-free } u, \quad \mathcal{L}[u] = 0 \iff u = \nabla^\perp g(z) = -g'(\cos \theta) \sin \theta \mathbf{e}_\phi$$

for some function $g : [-1, 1] \mapsto \mathbb{R}$. In other words, $\ker\{\mathcal{L}\}$ is the space of **longitude-independent zonal flows**. The authors also define a projection operator onto $\ker\{\mathcal{L}\}$

$$\Pi_{\ker\{\mathcal{L}\}} u := \frac{\left(\oint_{C(\theta)} u \cdot \mathbf{e}_\phi \right) \mathbf{e}_\phi}{\oint_{C(\theta)} \mathbf{e}_\phi \cdot \mathbf{e}_\phi} = \frac{1}{2\pi \sin \theta} \left(\oint_{C(\theta)} u \cdot \mathbf{e}_\phi \right) \mathbf{e}_\phi$$

where $\oint_{C(\theta)}$ is the line integral along the circle $C(\theta)$ at a fixed colatitude θ . Notice this actually defines an orthogonal projection in the Hilbert space $L^2(\mathbb{S}^2)$.

The key estimate (2) is proved in [6, Theorem 4.1] using spherical harmonics

$$Y_l^m(\phi, \theta) = N_l^m e^{im\phi} Q_l^m(\cos \theta) \text{ with } l = 0, 1, 2, \dots \text{ and } |m| \leq l.$$

Here, N_l^m is a normalization constant so that $\int_{\mathbb{S}^2} |Y_l^m|^2 = 1$.

First, rewrite any div-free velocity field u using stream function $u = \nabla^\perp \Psi$ where $\int_{\mathbb{S}^2} \Psi = 0$. Then, expand Ψ in terms of $\{Y_l^m\}$,

$$\Psi = \sum_{l=1}^{\infty} \sum_{m=-l}^l \psi_l^m Y_l^m.$$

Then, express operator \mathcal{L} as

$$\mathcal{L}[\nabla^\perp \Psi] = \nabla^\perp \sum_{l=1}^{\infty} \sum_{\substack{m=-l \\ m \neq 0}}^l \frac{-im}{l(l+1)} \psi_l^m Y_l^m \quad (12)$$

and the projection

$$\left(1 - \Pi_{\ker\{\mathcal{L}\}}\right)(\nabla^\perp \Psi) = \nabla^\perp \sum_{l=1}^{\infty} \sum_{\substack{m=-l \\ m \neq 0}}^l \psi_l^m Y_l^m. \quad (13)$$

The above two series reveal that spherical harmonics are eigenfunctions of \mathcal{L} . The ‘‘symbol’’ of \mathcal{L} is $\frac{-im}{l(l+1)}$, which has an ‘‘inverse’’ when restricted to the series for $\left(1 - \Pi_{\ker\{\mathcal{L}\}}\right)(\nabla^\perp \Psi)$ — note $m = 0$ is excluded from the sums in both (12) and (13). The conclusion of Theorem 4.1 in [6] is

$$\left\| u - \Pi_{\ker\{\mathcal{L}\}} u \right\|_{H^k} \leq \left\| \mathcal{L}[u] \right\|_{H^{k+2}}$$

which verifies the key assumption (2).

3. Shallow Water Equations on a Fast Rotating Sphere. For 2D *compressible* flows on the rotating sphere \mathbb{S}^2 , a very popular geophysical model is the Rotating Shallow Water (RSW) equations,

$$\partial_t h + \nabla_u h + h \operatorname{div} u = -\frac{1}{\varepsilon} \operatorname{div} u \quad (14)$$

$$\partial_t u + \nabla_u u = -\frac{1}{\varepsilon} \nabla h + \frac{z}{\varepsilon} u^\perp \quad (15)$$

where the Froude number (analogue to the Mach number) and Rossby number are both set as ε . The unknown h is understood as perturbation of height against the background rescaled to be 1. In other words,

$$\text{total height} = 1 + \varepsilon h$$

in this nondimensional setting. Note that in (15) we once again use $\frac{z}{\varepsilon}$ to represent meridional variation in the Coriolis parameter.

Note that results have been established concerning solution regularity of the above system and related ones in the fast rotating regime with $\varepsilon \ll 1$. Please refer to [3, 2] and references therein for further discussion.

The main result of this section confirms once again the zonal-flow pattern in planetary circulations.

Theorem 3.1. *Consider the RSW equations (14), (15) with a solution in $C([0, T]; H^k(\mathbb{S}^2))$ for $k \geq 3$. Then, there exist a function $f(\cdot) : [-1, 1] \mapsto \mathbb{R}$ and a universal constant C only depending on k s.t.*

$$\begin{aligned} & \left\| \frac{1}{T} \int_0^T u \, dt - \nabla^\perp f(z) \right\|_{H^{k-3}} + \left\| \frac{1}{T} \int_0^T h \, dt - f_1(z) \right\|_{H^{k-3}} \\ & \leq C\varepsilon \left(\frac{2M}{T} + M^2 \right). \end{aligned}$$

where constant $M := \max_{t \in [0, T]} \|(h, u)\|_{H^k}$ and $f_1(z)$ is uniquely determined by $f_1'(z) = -zf'(z)$ and $\int_{-1}^1 f_1(z) dz = 0$.

Thus, not only $\frac{1}{T} \int_0^T u \, dt$ can be approximated by a **longitude-independent zonal flow**, but also $\frac{1}{T} \int_0^T h \, dt$ can be approximated by a longitude-independent height field.

In the framework of (1) — (4), the main task is to identify \mathcal{L} , $\ker\{\mathcal{L}\}$, $\Pi_{\ker\{\mathcal{L}\}}$ and to establish the key estimate (2).

Apparently, by the RHS of (14), (15), we define

$$\mathcal{L}[(h, u)] := \begin{pmatrix} -\operatorname{div} u \\ -\nabla h + zu^\perp \end{pmatrix}$$

WLOG, also impose zero global mean condition on h

$$\int_{\mathbb{S}^2} h(t, x) \, dx = 0 \quad \text{for all times } t$$

since this condition is invariant in time thanks to the conservation law (14).

Examine definition (16) and obtain

$$(h, u) \in \ker\{\mathcal{L}\} \iff \begin{aligned} \operatorname{div} u &= 0 \quad \text{and} \end{aligned} \tag{16}$$

$$-\nabla h + zu^\perp = 0. \tag{17}$$

Now, $\ker\{\mathcal{L}\}$ will be identified in the following lemma. It came to us as a bonus that the velocity component of $\ker\{\mathcal{L}\}$ coincides exactly with $\ker\{\mathcal{L}\}$ in (12) of the previous section.

Lemma 3.2. *Consider height field h with zero global mean and velocity field u that is not necessarily div-free or curl-free. Assume they both have sufficient regularities. Then,*

$$(h, u) \in \ker\{\mathcal{L}\} \iff \text{there exists a scalar function } g(\cdot) : [-1, 1] \mapsto \mathbb{R} \text{ s.t.}$$

$$u = \nabla^\perp g(z), \quad \text{and} \quad h = g_1(z)$$

where $g_1(z)$ is uniquely determined by $g_1'(z) = -zg'(z)$ and $\int_{-1}^1 g_1(z) dz = 0$.

In the following proof, we apply Hodge decomposition to vector fields on \mathbb{S}^2

$$u = u_{\text{irr}} + u_{\text{inc}} := \nabla \Delta^{-1} \operatorname{div} u + \nabla^\perp \Delta^{-1} \operatorname{curl} u.$$

The exact definitions and relevant properties can be found in [6, Appendix A].

Proof. Apply Hodge decomposition to (17)

$$-\nabla h + \nabla \Delta^{-1} \operatorname{div} (zu^\perp) + \nabla^\perp \Delta^{-1} \operatorname{curl} (zu^\perp) = 0.$$

Due to the uniqueness of Hodge decomposition, both the irrotational and incompressible parts on the LHS vanish. Together with (16), we arrive at

$$(h, u) \in \ker\{\mathcal{L}\} \iff$$

$$\operatorname{div} u = 0, \quad \nabla^\perp \Delta^{-1} \operatorname{curl} (zu^\perp) = 0 \quad \text{and} \tag{18}$$

$$-\nabla h + \nabla \Delta^{-1} \operatorname{div} (zu^\perp) = 0. \tag{19}$$

Observe that (18) coincides with the definition of \mathcal{L} in (12) and the div-free condition posed for the incompressible Euler equations. Therefore, by (12), we have

$$(18) \iff u = \nabla^\perp g(z).$$

For the h component, (19) implies ∇h is the curl-free component in the Hodge decomposition of zu^\perp ; but the above condition makes $zu^\perp = -z\nabla g(z)$ automatically curl-free. Therefore, for $(h, \nabla^\perp g(z)) \in \ker\{\mathcal{L}\}$,

$$\nabla h = -z\nabla g(z) \iff h = g_1(z) \text{ with } g_1'(z) = -zg'(z).$$

And, because the h component is always of zero global mean, so is $g_1(z)$. \square

Now, we introduce a projection $\Pi_{\ker\{\mathcal{L}\}}$, not necessarily orthogonal projection, based on the projection defined in (12) from the previous section.

Lemma 3.3. (*Characterization of $\Pi_{\ker\{\mathcal{L}\}}$*) Consider height field h with zero global mean and velocity field u that is not necessarily div-free. Assume enough regularities for both. Then, the following defines a projection operator associated with $\ker\{\mathcal{L}\}$,

$$\begin{aligned} \Pi_{\ker\{\mathcal{L}\}}(h, u) &= (\Delta^{-1} \operatorname{div}(z\tilde{u}^\perp), \tilde{u}) \\ \text{where } \tilde{u} &:= \frac{\left(\oint_{C(\theta)} u_{\text{inc}} \cdot \mathbf{e}_\phi \right) \mathbf{e}_\phi}{\oint_{C(\theta)} \mathbf{e}_\phi \cdot \mathbf{e}_\phi} \end{aligned} \quad (20)$$

Here, $u_{\text{inc}} = \nabla^\perp \Delta^{-1} \operatorname{curl} u$ is the div-free component in the Hodge decomposition of u ; and $\oint_{C(\theta)}$ is the line integral along the circle $C(\theta)$ at a fixed colatitude θ .

Remark 4. We can also use $u = u_{\text{irr}} + u_{\text{inc}}$ instead of u_{inc} in the $\oint_{C(\theta)}$ integral of (20), knowing that Stokes Theorem guarantees the circulation of u_{irr} over any closed path vanishes.

Straightforward calculation can show that $\Pi_{\ker\{\mathcal{L}\}}$ is a projection, i.e. satisfies

$$\Pi_{\ker\{\mathcal{L}\}} \circ \Pi_{\ker\{\mathcal{L}\}} = \Pi_{\ker\{\mathcal{L}\}},$$

but it is not an L^2 -orthogonal projection anymore. Although the L^2 -orthogonal projection does exist in a Hilbert space setting, it is unclear that such a projection satisfies the estimate (2) in any H^k spaces, especially those admitting classical solutions.

On the other hand, we state and prove below that projection $\Pi_{\ker\{\mathcal{L}\}}$ defined in Lemma 3.3 satisfies the key estimate (2) for any H^k spaces, as needed for (1) – (4). A key motivation is that projection defined in (20) coincides with the projection defined in (12) from previous section; this allows one to utilize estimate (14) from previous section as well,

$$\|u_{\text{inc}} - \tilde{u}\|_{H^k} \leq \|\nabla^\perp \Delta^{-1} \operatorname{curl}(zu_{\text{inc}}^\perp)\|_{H^{k+2}}. \quad (21)$$

Theorem 3.4. For any height field $h \in H^k(\mathbb{S}^2)$ with zero global mean and vector field $u \in H^k(\mathbb{S}^2)$ not necessarily div-free or curl-free, and for $\Pi_{\ker\{\mathcal{L}\}}$ defined in (20), there exists a universal constant $C = C(k)$ s.t.

$$\left\| (h, u) - \Pi_{\ker\{\mathcal{L}\}}(h, u) \right\|_{H^k} \leq C \|\mathcal{L}[(h, u)]\|_{H^{k+2}}$$

Proof. WLOG, assume

$$\|\mathcal{L}[(h, u)]\|_{H^{k+2}} = 1. \quad (22)$$

By Lemma 3.3, we need to estimate

$$(h, u) - \Pi_{\ker\{\mathcal{L}\}}(h, u) = (h - \Delta^{-1} \operatorname{div}(z\tilde{u}^\perp), u - \tilde{u}) \quad (23)$$

with \tilde{u} defined in (20).

We will repeatedly use the fact that Hodge decomposition of vector fields on \mathbb{S}^2 satisfies

$$\|u_{\text{irr}}\|_{H^k} \leq C \|\operatorname{div} u\|_{H^{k-1}}, \quad (24)$$

$$\|u_{\text{inc}}\|_{H^k} \leq C \|\operatorname{curl} u\|_{H^{k-1}}, \quad (25)$$

which can be proved by standard routine.

- H^k Estimates of the velocity component $(u - \tilde{u}) = u_{\text{irr}} + (u_{\text{inc}} - \tilde{u})$. First, by the definition of \mathcal{L} in (16), and the normalization (22),

$$\| -\nabla h + zu^\perp \|_{H^{k+2}} + \| \operatorname{div} u \|_{H^{k+2}} \leq 1. \quad (26)$$

Immediately, estimate on the curl-free part u_{irr} is obtained by using the above inequality and (24)

$$\| u_{\text{irr}} \|_{H^{k+3}} \leq C \| \operatorname{div} u \|_{H^{k+2}} \leq C. \quad (27)$$

Now, we estimate the div-free part $(u_{\text{inc}} - \tilde{u})$. Apply Hodge decomposition on $-\nabla h + zu^\perp$ from the LHS of (26)

$$-\nabla h + zu^\perp = -\nabla h + \nabla \Delta^{-1} \operatorname{div} (zu^\perp) + \nabla^\perp \Delta^{-1} \operatorname{curl} (zu^\perp)$$

and use (24), (25), (26) to obtain

$$\| -\nabla h + \nabla \Delta^{-1} \operatorname{div} (zu^\perp) \|_{H^{k+2}} \leq C, \quad (28)$$

$$\| \nabla^\perp \Delta^{-1} \operatorname{curl} (zu^\perp) \|_{H^{k+2}} \leq C. \quad (29)$$

Apply (21), obtained in the previous section, triangle inequality and (27), respectively,

$$\begin{aligned} & \| u_{\text{inc}} - \tilde{u} \|_{H^k} \\ & \leq \| \nabla^\perp \Delta^{-1} \operatorname{curl} (zu_{\text{inc}}^\perp) \|_{H^{k+2}} \\ & \leq \| \nabla^\perp \Delta^{-1} \operatorname{curl} (zu^\perp) \|_{H^{k+2}} + \| \nabla^\perp \Delta^{-1} \operatorname{curl} (zu_{\text{irr}}^\perp) \|_{H^{k+2}} \\ & \leq \| \nabla^\perp \Delta^{-1} \operatorname{curl} (zu^\perp) \|_{H^{k+2}} + C. \end{aligned}$$

Since a bound for $\nabla^\perp \Delta^{-1} \operatorname{curl} (zu^\perp)$ has been established in (29), we arrive at

$$\| u_{\text{inc}} - \tilde{u} \|_{H^k} \leq C.$$

With estimate (27), we obtain

$$\| u - \tilde{u} \|_{H^k} \leq C. \quad (30)$$

- H^k Estimates of the height component $(h - \Delta^{-1} \operatorname{div} (z\tilde{u}^\perp))$. First, notice that the zero-global-mean condition (16) and the Poincaré inequality imply

$$\| h - \Delta^{-1} \operatorname{div} (z\tilde{u}^\perp) \|_{H^k} \leq C \| \nabla h - \nabla \Delta^{-1} \operatorname{div} (z\tilde{u}^\perp) \|_{H^{k-1}}. \quad (31)$$

Then, by the triangle inequality, (28), (30), respectively,

$$\begin{aligned} & \text{RHS of (31)} \\ & \leq \| \nabla h - \nabla \Delta^{-1} \operatorname{div} (zu^\perp) \|_{H^{k-1}} + \| \nabla \Delta^{-1} \operatorname{div} (z(u - \tilde{u})^\perp) \|_{H^{k-1}} \\ & \leq C. \end{aligned}$$

Together with (30) and (23), this finishes the proof. \square

4. Compressible Euler Equations in Bounded Domain. Consider compressible Euler equations in a simply connected, bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) with smooth boundary $\partial\Omega$ for the unknown pair (ρ, u) ,

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) + \frac{\nabla \cdot u}{\varepsilon} = 0 \\ \partial_t u + \nabla_u u + \frac{p'(1 + \varepsilon \rho)}{1 + \varepsilon \rho} \frac{\nabla \rho}{\varepsilon} = 0 \end{cases} \quad (32)$$

$$u \cdot \mathbf{n} \Big|_{\partial\Omega} = 0, \quad (33)$$

$$\int_{\Omega} \rho dx = 0. \quad (34)$$

Here, $0 < \varepsilon \ll 1$ denotes the Mach number. u denotes the velocity field and ρ denotes the rescaled density perturbation against the background 1 so that

$$\text{total density} = 1 + \varepsilon\rho.$$

The equation of state is

$$\text{pressure} = p(1 + \varepsilon\rho)$$

with $p(\cdot) \in C^\infty$, $p(1) = 0$, $p'(1) = 1$. Solid-wall boundary condition (33) is imposed on the velocity field u where $\mathbf{n} = \mathbf{n}(x)$ denotes the outward normal at $x \in \partial\Omega$. Also, without loss of generality, impose (34) since $\int_{\Omega} \rho dx$ is conserved in the dynamics.

In [5, Theorem 2.5], the author proves the following theorem (reformulated to suit this paper)

Theorem 4.1. *Consider the Euler equations (32) — (34) with solution in $C([0, T], H^m(\Omega))$. Then, there exists a splitting of the velocity field $u = u_{\text{inc}} + u_{\text{irr}}$ into incompressible part u_{inc} and irrotational part u_{irr} , so that*

$$\left\| \frac{1}{T} \int_0^T (\rho, u_{\text{irr}}) dt \right\|_{H^m(\Omega)} \leq C\varepsilon \left(1 + \frac{1}{T} \right).$$

Here, C only depends on $\max_{t \in [0, T]} \|U\|_{H^m(\Omega)}$ and m .

This result validates the incompressible approximation of low-Mach-number flows in a bounded domain. To mention only a few earliest work, we note [4, 7, 8, 15, 16] for well-prepared initial data on domains without boundary condition. Note that, with ill-prepared initial data and solid-wall boundary condition, there is no dissipative or dispersive mechanism to control the fast acoustic waves. It is once again the time-averaging process that bounds the fast acoustic waves in terms of Mach number ε .

We now recap the proof for this theorem, in the abstract framework of (1) — (4). The main task is to identify \mathcal{L} , $\ker\{\mathcal{L}\}$, $\Pi_{\ker\{\mathcal{L}\}}$ and to establish the key estimate (2).

Start with rewriting (32) in the form of (1),

$$\partial_t \begin{pmatrix} \rho \\ u \end{pmatrix} = \frac{1}{\varepsilon} \mathcal{L} \begin{pmatrix} \rho \\ u \end{pmatrix} + F,$$

where

$$\mathcal{L} \begin{pmatrix} \rho \\ u \end{pmatrix} := - \begin{pmatrix} \nabla \cdot u \\ \nabla \rho \end{pmatrix},$$

and the nonlinear term

$$F := \begin{pmatrix} \nabla \cdot (\rho u) \\ \nabla_u u + \frac{1}{\varepsilon} \left(\frac{p'(1 + \varepsilon\rho)}{1 + \varepsilon\rho} - 1 \right) \nabla \rho \end{pmatrix}.$$

Note that the dependence of F on ε is not singular, i.e., we can bound F uniformly in $\varepsilon \ll 1$.

Due to the constraints (33), (34), we define a solution space as

$$X^m := \left\{ \begin{pmatrix} \rho \\ u \end{pmatrix} \in H^m(\Omega) \mid u \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \text{ and } \int_{\Omega} \rho dx = 0 \right\}.$$

Note that $\ker\{\mathcal{L}\} \cap X^m = \left\{ \begin{pmatrix} 0 \\ u \end{pmatrix} \in H^m(\Omega) \mid u \cdot \mathbf{n}|_{\partial\Omega} = 0, \nabla \cdot u = 0 \right\}$. Then, we choose

$\Pi_{\ker\{\mathcal{L}\}}$ as the L^2 -orthogonal-projection onto $\ker\{\mathcal{L}\} \cap X^m$

$$\begin{aligned} \text{For } u \in X^m, \quad v = \Pi_{\ker\{\mathcal{L}\}} u &\iff \\ v \in \ker\{\mathcal{L}\} \cap X^m \text{ and } (u - v) &\text{ is } L^2\text{-orthogonal to } \ker\{\mathcal{L}\}. \end{aligned} \quad (35)$$

Standard elliptic PDE theory guarantees the following lemma.

Lemma 4.2. *Operator defined in (35) satisfies,*

$$\text{for any } \begin{pmatrix} \rho \\ u \end{pmatrix} \in X^m, \quad \Pi_{\ker\{\mathcal{L}\}} \begin{pmatrix} \rho \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ u - \nabla\phi \end{pmatrix}$$

with ϕ uniquely solving

$$\begin{cases} \Delta\phi = \nabla \cdot u, \\ \nabla\phi \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \int_{\Omega} \phi = 0 \end{cases} \quad (36)$$

Finally, to prove the key estimate (2), it suffices to show

$$\|\rho\|_{H^m} + \|\nabla\phi\|_{H^m} \leq C(\|\nabla \cdot u\|_{H^{m-1}} + \|\nabla\rho\|_{H^{m-1}}).$$

The estimate on ρ is easily obtained by using the Poincaré inequality and (34). The estimate on $\nabla\phi$ is obtained by applying standard elliptic PDE theory on the Poisson equation (36) with Neumann boundary condition.

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