Math 471 Numerical Methods

Homework 6. Due: Aug. 13 (80'+10' bonus)

Problems requiring coding are marked with $*$.

Please do some editing to minimize the pages of your print-outs. Thanks!

1. (7') Let $r(x)$ be certain type of interpolation (not necessarily a single polynomial) that is already computed for data set $\{x_i, y_i\}_{i=0}^n$ and let $s(x)$ be another interpolation for $\{x'_j, y'_j\}_{j=0}^m$. All x_i's and x'_j's are distinct. Find two polynomials $p(x)$ and $q(x)$ so that $p(x)r(x) + q(x)s(x)$ interpolates $\{x_i, y_i\}_{i=0}^n \cup \{x'_j, y'_j\}_{j=0}^m$.

2. (10') Find the 3rd degree polynomial interpolation $p(x)$ by hand to fit data set

$$(0, -1), (2, 2), (1, -2), (4, 0)$$

Use both the Lagrange form and the Newton form. Then, show that these two forms are essentially the same.

3. (15') A city is planning to construct a beltway surrounding its downtown area. It is supposed to pass through n points of interests

$$\{r_i, \theta_i\}_{i=1}^n \quad \text{with } 0 < \theta_1 < \theta_2 < ... < \theta_n < 2\pi \tag{1}$$

where polar coordinates are used with the origin fixed at the center of the city. The cubic spline interpolation is to be used to find a smooth C^2 function $r = s(\theta)$ for $\theta \in [0, 2\pi]$. Because of the circular nature of the beltway, $s(\theta)$ has be to 2π-periodic

$$s(\theta) = s(\theta + 2\pi) \quad \text{for any given } \theta$$

and thus

$$s(\theta) \text{ can be extended for any } \theta \in [-\infty, \infty]$$

(a) A duplicated point $(r_{n+1}, \theta_{n+1}) = (r_1, \theta_1 + 2\pi)$ needs to be included in the interpolation. Then, the n cubic polynomials to be found are

$$s_1(\theta) \text{ interpolating } (r_1, \theta_1) \text{ and } (r_2, \theta_2)$$

$$s_2(\theta) \text{ interpolating } (r_2, \theta_2) \text{ and } (r_3, \theta_3)$$

....

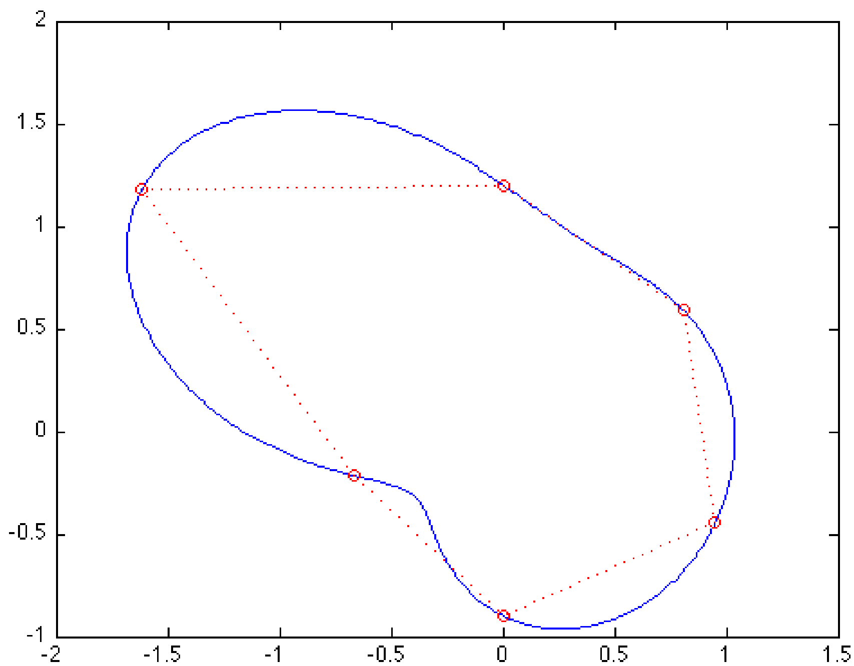
$s_n(\theta)$ interpolating (r_n, θ_n) and (r_{n+1}, θ_{n+1}).

Two boundary conditions are to be imposed at θ_1 and θ_n

$$s_1'(\theta_1) = s_n'(\theta_{n+1}), \quad s_1''(\theta_1) = s_n''(\theta_{n+1}). \tag{2}$$

Explain briefly why these conditions are necessary to guarantee the interpolation with continuous second order derivatives at **all** data points.

(b) Below is an example of connecting 6 points of interests (in red color). Mark on the graph points $\{r_i, \theta_i\}_{i=1}^7$ and the segments where the curve is determined by one of the cubic polynomials $r = s_i(\theta)$, $i = 1, 2, ..., 6$.



(c) If the convenient formula

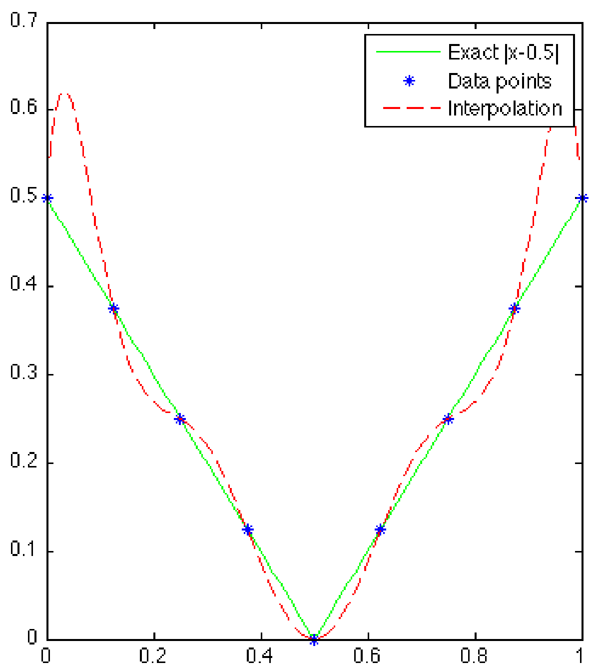
$$s_i(\theta) = \frac{a_i}{6h_i}(\theta_{i+1} - \theta)^3 + \frac{a_{i+1}}{6h_i}(\theta - \theta_i)^3 + \left(r_i - \frac{a_i h_i^2}{6}\right)\frac{\theta_{i+1} - \theta}{h_i} + \left(r_{i+1} - \frac{a_{i+1} h_i^2}{6}\right)\frac{\theta - \theta_i}{h_i} \tag{3}$$

for $\theta \in [\theta_i, \theta_{i+1}]$ and $n_1 \le i \le n_2$

what are n_1 and n_2 for the range of i, and what are h_{n_1} and h_{n_2}? And, why do we have to impose $a_1 = a_{n+1}$ to reflect one of the boundary conditions in (2)?

(d) Construct a linear system to find the cubic spline interpolation for the general case (1). Is this system n-by-n, (n-1)-by-(n-1) or (n+1)-by-(n+1)? Pay special attention to the first and last equations in your system that will not be exactly but very close to a tri-diagonal system.

4. * (15') Let $f(x) = |x - 0.5|$. Interpolate at $x_i = i/8$ for $0 \le i \le 8$. Then, evaluate the resulting polynomial at $x_{i+0.5} = (i + 0.5)/8$ for $0 \le i \le 7$ and the associated erros. Plot the errors against $x_{i+0.5}$ (so you want to use `plot(xdata,ydata)` in Matlab) and explain what you observe using the formula for **error bound**.
* Use either Newton or Lagrange form as you like. You may even use the 9-by-9 linear system for the coefficients in $a_0 + a_1 x + a_2 x^2 + ... + a_8 x^8$.
* The following graph is a comparison of the exact $f(x)$, the data points at $\{x_i\}$ and the 8-th degree polynomial that interpolates between the data points. Use it to check your graph of errors.



5. * (15'+5') Extrapolate centered finite difference for 1st derivatives

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h} \stackrel{def}{=} D(0)f(x)$$

for $f(x) = sin(x^2)$ at $x = \pi$. Use $h = 1/2, 1/4, 1/8, 1/16$. Denote this extrapolation by $D(1)f(x)$.

(a) Write down the formula for $D(1)f(x)$ and also express it in terms of 5 points $f(x-2h), f(x-h), f(x), f(x+h), f(x+2h)$.

(b) Make observations on the errors from this extrapolation: what is the order of accuracy now?

(c) Use the above information on order of convergence to extrapolate again, denoted by $D(2)f(x)$.

(d) Create a table with 4 columns: one for h, one for the error from the original centered difference, one for the errors from the first extrapolation and one for the errors from the second extrapolation. Part of the table is given below

h	f'-D(0)f	f'-D(1)f	f'-D(2)f
0.5000	-5.6717	0	0
0.2500	-2.1756	-1.0102	0
0.1250	-0.6040	*******	*******
0.0625	*******	-0.0053	-0.0003

(e) What is the order of accuracy for this second extrapolation?

(f) (bonus 5') * Reduce the size h in the previous problem so that round-off error becomes dominant. Briefly explain what you observe in the errors for very small h's.

6. (18'+5') Let n be an even, positive integer. Let $I_n(f)$ denote the closed Newton-Cotes quadrature rule with $n+1$ abscissas

$$I_n(f) = \sum_{i=0}^{n} w_i f(x_i)$$

Here, w_i are the weights calculated using abscissas $x_0, x_1, x_2 ... x_n$.

(a) Write down the formula for calculating $w_i^{(1)}$ if $x_i = i/n$.

(b) Write down the formula for calculating $w_i^{(h)}$ if $x_i = ih/n$ with a fixed h and show that $w_i^{(h)} = hw_i^{(1)}$

(c) What is the error bound for part (b) in terms of n, h and derivatives of f?

(d) Construct a **composite** quadrature rule using $I_n(f)$ to approximate

$$\int_0^{kh} f(x)dx = \int_0^h fdx + \int_h^{2h} fdx + ... \int_{(k-1)h}^{kh} fdx.$$

(e) (bonus 5') What is the error bound for the quadrature from part (d) in terms of k, n, h and derivatives of f?