# Math 471 Numerical Methods

Homework 4. Due: Jul. 30

Problems requiring coding are marked with $*$.

**Please do some editing to minimize the pages of your print-outs. Thanks!**

1. * Determine the optimal value of $\omega$ in the SOR method for the following positive definite, tridiagonal system. Then, calculate the corresponding spectral radius of SOR method. Finally, perform the SOR method with this optimal $\omega$ and stop the iteration when the infinity norm of $\vec{x}_{k+1} - \vec{x}_k$ is less than $5 * 10^{-6}$. Do 3 runs with the initial guess being $(0 \quad 0 \quad 0)^T$, $(1 \quad 2 \quad 3)^T$ and $[300 \quad 400 \quad 500]^T$. How many iterations are used to achieve this accuracy? Make sure you use the absolute value when calculating infinity norms for vectors. (Answer: 6$\sim$7 iterations or so.)

$$\begin{aligned} 4x_1 - x_2 \qquad &= 2 \\ -x_1 + 4x_2 - x_3 &= 4 \\ -x_2 + 4x_3 &= 0 \end{aligned}$$

2. Show that for an iteration scheme with invertible $B$,

$$\vec{x}_{k+1} = B\vec{x}_k + \vec{c},$$

the following ratio also converges to the spectral radius,

$$\lim_{k\to\infty} \frac{\|\vec{x}_{k+1} - \vec{x}_k\|}{\|\vec{x}_k - \vec{x}_{k-1}\|} = \rho(B).$$

You may use the fact that, for any nonzero vector $\vec{y}$ and invertible matrix $T$,

$$\lim_{k\to\infty} \frac{\|T^{k+1}\vec{y}\|}{\|T^k\vec{y}\|} = \rho(T)$$

3. Consider a 2nd order ODE of boundary value problem

$$(p(x)u'(x))' = f(x), \quad u(0) = a, \;\; u(L) = b,$$

where $p(x), f(x)$ are given functions and $u(x)$ is the unknown. Use a uniform grid $x_i = i * h$ with $h = L/n$ to discretize the ODE into a $(n-1)\times(n-1)$ linear system

$$A\vec{x} = \vec{b}.$$

The requirement is that $A$ has to be **symmetric** and tridiagonal, which is of great convenience in practice. You may follow these steps,

(a) The centered difference for approximating $(p(x)u'(x))'$ is

$$(pu')'\big|_{x=x_i} \approx \frac{(pu')\big|_{x=x_{i+1/2}} - (pu')\big|_{x=x_{i-1/2}}}{h}$$

whereas the centered difference for approximating $(pu')$ is

$$(pu')\big|_{x=x_{i+1/2}} \approx p(x_{i+1/2})\frac{u(x_{i+1}) - u(x_i)}{h}.$$

Similar formula works for $(pu')'\big|_{x=x_{i-1/2}}$.

(b) Show that for a generic i-th row in $A$, the 3 nonzero entries are

$$\frac{1}{h^2}p_{i-1/2},\ -\frac{1}{h^2}p_{i-1/2} - \frac{1}{h^2}p_{i+1/2},\ \frac{1}{h^2}p_{i+1/2}.$$

(c) Construct the entire $A$ so that it is symmetric and tridiagonal. Pay attention to the first and last row.

4. If the boundary condition in the previous problem is changed to

$$u(0) = a, \quad u(L) + 2u'(L) = b$$

we have to use certain difference scheme on the right boundary.

(a) Describe the changes you need to make in the system obtained from Problem 3 if the following finite difference is used for $u'(L)$,

$$u'(L) \approx \frac{u(L) - u(L-h)}{h}.$$

(b) Describe the changes you need to make in the system obtained from Problem 3 if the following finite difference is used for $u'(L)$,

$$u'(L) \approx \frac{u(L-2h) - 4u(L-h) + 3u(L)}{2h}.$$

(c) (optional) Show that the truncation error in (a) is $O(h)$ whereas in (b) is $O(h^2)$.

5. Construct a discretization for the Poisson equation

$$u_{xx} + u_{yy} + \alpha(x,y)u_x + \beta(x,y)u_y + \gamma(x,y)u = Q(x,y),$$

$$u(x,y) = 0 \quad \text{on all sides of } [0,L]\times[0,H].$$

Here, $u(x,y)$ is the unknown. Make sure the coefficient matrix $A$ is block-tridiagonal. Give convincing descriptions of the blocks. You may use $h$ as the grid size for both $x$ and $y$ direction.