Math 471 Numerical Methods

Homework 3. Due: Jul. 23

Problems requiring coding are marked with $*$.

Please do some editing to minimize the pages of your print-outs. Thanks!

1. Generate an $(2m+1)$-by-$(2m+1)$, lower triangular matrix L using Matlab code `L=diag(1:(2*m+1))+6*diag(ones(2*m,1),-1)`. It's inverse can be found using code `inv(L)`.

   (a) Let $m = 5$ and run the above codes. Descibe the main difference you see in the pattern of nonzero entries in L and L^{-1}.

   (b) What are the operation counts for two (theoretically) equivalent approaches of finding $\vec{y}$: $L\vec{y} = \vec{b}$ and $\vec{y} = L^{-1}\vec{b}$? Please use notation $O(m^k)$ with k specified and indicate which approach is computationally faster for general m's.

2. Define a lower triangular matrix

$$M = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & m_{j+1,j} & \ddots & & \\ & \vdots & & \ddots & \\ & m_{n,j} & & & 1 \end{pmatrix}$$

where diagonal entries are 1 and all other entries are zero except the lower part of column j. Define a permutation matrix

$$P = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & p_{k,k} = 0 & & p_{k,n} = 1 \\ & & & \ddots & \\ & & p_{n,k} = 1 & & p_{n,n} = 0 \end{pmatrix}.$$

Assume (very important!) $k > j$. Find the matrix $\hat{M}$ in the equation (with the help of the "rule of thumb": left-multiplication $\Longleftrightarrow$ row operation; right-multiplication $\Longleftrightarrow$ column operation)

$$PM = \hat{M}P.$$

Note: a more general version of this equation allows us to "move" all the permutation matrices out of the sequence of matrices generated from Guassian Elimination with pivoting.

3. Consider a 10-by-10 matrix A. (Note the only nonzeros are: diagonal entries of 5's and $a_{2,8} = 2$, $a_{3,2} = -2$, $a_{9,1} = 3$, $a_{10,2} = -1$.)

5	0	0	0	0	0	0	0	0	0
0	5	0	0	0	0	0	2	0	0
0	-2	5	0	0	0	0	0	0	0
0	0	0	5	0	0	0	0	0	0
0	0	0	0	5	0	0	0	0	0
0	0	0	0	0	5	0	0	0	0
0	0	0	0	0	0	5	0	0	0
0	0	0	0	0	0	0	5	0	0
3	0	0	0	0	0	0	0	5	0
0	-1	0	0	0	0	0	0	0	5

   (a) Do a LU factorization by hand on A and then use Matlab code `[L U]=lu(A)` to verify your result;

   (b) Solve $A\vec{x} = \vec{b}$ by hand using the above LU decomposition for $b = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)^T$. Do NOT use the Gaussian elimination.

4. Use the Doolittle method, simplied for tri-diagonal matrix, to perform a direct LU factorization by hand on the following matrix. Make sure the diagonal entries of L (instead of U as in the Crout method) are 1.

-2	2	0	0
2	-3	-3	0
0	-2	-4	1
0	0	6	4

5. Consider the following linear system

$$\begin{cases} 6x - 2y + 3z = -5 \\ -5x + 10y - 2z = -15 \\ 4x - y + 6z = -24 \end{cases}$$

for which the exact solution is $x = 1$, $y = -2$, $z = -5$.

(a) * Write down the Jacoci method in mathematical terms. You may use numbers explicitly in your scheme, but it may give you a messier form than a generic formulation for n-by-n matrices. Then, implement it and test your code in Matlab or other programming software you would like.

(b) * Do the same thing with the Guass-Seidel method.

(c) Once you make sure both of your codes converge to the right answer, do the following calculation to both methods,

   i) Run your codes for 31 iterations, with initial guess $x_0 = (0, 0, 0)^T$. Let e_k denote the error from running the k-th iteration. Compute $\|e_{11}\|_\infty/\|e_{10}\|_\infty$, $\|e_{21}\|_\infty/\|e_{20}\|_\infty$, $\|e_{31}\|_\infty/\|e_{30}\|_\infty$. What is the order of convergence you observe? What is the asymptotic constant?

   ii) Write down the associated iteration matrix B for the iteration scheme and calculate its numeric value (with a software if necessary). What is $\|B\|_\infty$? Is the error relation

$$\|e_{k+1}\|_\infty \le \|B\|_\infty \|e_k\|_\infty$$

   consistent with i)? Briefly explain.

6. In solving linear systems, one can use fixed point iteration that involves more than one previous values,

$$x_{k+1} = Bx_k + Cx_{k-1} + d.$$

(a) Show that if this scheme converge, it converges as

$$\lim_{k\to\infty} x_k = (I - B - C)^{-1} d.$$

(b) Show that if $0 < r = \|B\| + \sqrt{\|C\|} < 1$ in certain norm, then the scheme converges in the same norm at rate $O(r^k)$.

Hint: if a positive sequence $\{y_k\}$ satisfies $y_{k+1} \le by_k + cy_{k-1}$ for $b > 0$ and $c > 0$, then one can use induction to show

$$y_k \le \gamma \left(b + \sqrt{c}\right)^k$$

for certain constants γ.