Math 471 Numerical Methods

Homework 2. Due: Jul. 16

Problems requiring coding are marked with $*$.

1. (another version of the convergence theorem for fixed point iteration) Consider scheme

$$x_{n+1} = g(x_n),$$

where $g(x)$, $g'(x)$ are continuous for $x \in [a, b]$. Let

$$0 < k = \max_{x\in[a,b]} |g'(x)| < 1.$$

Also, choose $x_0 = (a + b)/2$. Prove that $\{x_n\}$ converges to a fixed point p of $g(x)$ if

$$\frac{|g(x_0) - x_0|}{1 - k} \le \frac{b - a}{2}. \tag{1}$$

Hint: it suffices to show that all $x_n \in [a, b]$ (why?). To prove this, first show that $x_1 = g(x_0) \in [a, b]$ using assumption (1). Then, study the successive difference $x_{n+1} - x_n$ **in terms of** $x_n - x_{n-1}$ by using the iterative relation $x_{n+1} = g(x_n)$ **and** $x_n = g(x_{n-1})$. This relation should resemble the one discussed in class $x_{n+1} - p = g'(\xi)(x_n - p)$. Once you express $x_{n+1} - x_n$ in terms of $x_1 - x_0$, the rest is just sum up $(x_{n+1} - x_n) + (x_n - x_{n-1}) + ... + (x_1 - x_0) = x_{n+1} - x_0$ and show that it does not exceed $(b - a)/2$. Since x_0 is the middle point of $[a, b]$, such estimate should lead to $x_{n+1} \in [a, b]$.

2. Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show that, if $ad - bc \neq 0$, then

$$M^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

3. Use Gaussian elimination to solve the following system **by hand**.

$$\begin{aligned} 2x_1 - x_2 + x_3 &= -1 \\ 4x_1 + 2x_2 + x_3 &= 4 \\ 6x_1 - 4x_2 + 2x_3 &= -2 \end{aligned}$$

4. * Combine Gaussian elimination and partial pivoting to find the inverse of the following matrix. This time, you may use the code provided in lecture notes but have to write the codes for the subroutine `findmax` and `row_op1` that are used in the lecture notes.

$$\begin{pmatrix} 10^{-14} & 1/2 & 1/3 & 1/4 \\ 1/5 & 1/6 & 10^{-14} & 1/7 \\ 1/7 & 10^{-14} & 1/6 & 1/5 \\ 1/4 & 1/3 & 1/2 & 10^{-14} \end{pmatrix}$$

5. Use Gaussian elimination **without** pivoting to find the inverse of the same matrix as given in problem 4. You may use the code provided in lecture notes. Do you observe any difference between the results from 4 and 5? Which one is more accuate? And why?

6. * Write a code that takes an upper triangular matrix U and right hand side $\vec{c} = (c_1, c_2, ..., c_n)^T$ as the input arguments and outputs the solution to $U\vec{x} = \vec{c}$. Do NOT use any row operation or compute the actual inverse U^{-1} but instead use the backward substitution by its original meaning. What is the operation count as opposed to using row operations? You may use the $O(\cdot)$ notation.

7. When using partial pivoting with Gaussian elimination, what is the most possible operation count to reduce an $n - by - n$ matrix to an upper triangular one (i.e. echelon form)? It is enough to specify the constant C and α if the operation count is $Cn^\alpha + Dn^{\alpha-1} + ...$ Does the partial pivoting change the $O(\cdot)$ nature of the operation count for Guassian elimination?

8. Let $A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$ and $\mathbf{x} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$. Find $\|\mathbf{x}\|_\infty$, $\|\mathbf{x}\|_2$, $\|A\|_\infty$, $\|A\|_2$ by hand.