Math 471 Numerical Methods

Homework 1. Due: Jul. 9

Computational problems are marked with $*$.

1. Show that if a sequence $\{x_n\}$ converges with order 1 and asymptotic constant $\lambda > 0$, then it converges at rate $O((\lambda+\varepsilon)^n)$ for any $\varepsilon > 0$. Hint: 1. only need to show $|x_n - x_\infty| \le C(\lambda+\varepsilon)^n$ for sufficiently large n; 2. use the definition of limit.

2. What is the convergence rate of the geometric series $1/3 + 1/3^2 + 1/3^3 + ...$ towards $1/2$? What is the order of convergence?

3. Given two functions

$$f(x) = 1 - \sin x, \quad g(x) = \frac{\cos^2 x}{1 + \sin x}.$$

   (a) Verify $f(x) = g(x)$.

   (b) Which function should be used for computations when x is near $\pi/2$? Why?

   (c) Which function should be used for compuations when x is near $3\pi/2$? Why?

   (d*) Evaluate $1 - \sin(\frac{\pi}{2} + 2^{-30})$ to 4 digits. Include your code.

4. The centered difference for computing derivatives of $f(x)$ is

$$f'(x) \approx D_0 f(x) := \frac{f(x+h) - f(x-h)}{2h}.$$

   (a) Use Taylor series to show that the truncation error is of order $O(h^2)$.

   (b*) Let $f(x) = e^x$. Create a table with 4 columns h, $D_0 f(1)$, $D_0 f(1) - f'(1)$, $\dfrac{D_0 f(1) - f'(1)}{h^2}$. Fill in the table for $h = 10^{-2}, 10^{-3}, 10^{-4}$ and $h = 10^{-12}, 10^{-13}, 10^{-14}$. Include your code.

5. Consider $f(x) = x^3 - 6$. Since $f(1) < 0$ and $f(2) > 0$, it follows that $f(x)$ has a root p in the interval $[1, 2]$.

   (a*) By using the bisection method, compute an approximate root in $[1, 2]$ with 4 correct digits. How many iterations does your program require for such accuracy?

(b*) By using the Newton's method with initial guess $x_0 = 1.5$, compute an approximate root in $[1, 2]$ with 4 correct digits. How many iterations does your program require for such accuracy?

(c) Explain why iteration scheme $x_{n+1} = x_n - f(x_n)$ won't work for this problem no matter how carefully the initial guess x_0 is choosen.

(d) Explain why iteration scheme $x_{n+1} = x_n - f(x_n)/6$ may work for this problem so long as the initial guess x_0 is carefully choosen.