Math 454 HW 6

Due: Dec 10 at noon

1. (**Wave equation with source**.) Consider a $\pi \times \pi$ vibrating membrane. It is fixed at the edges $x = 0$ and $x = \pi$, and is free to move at the edges $y = 0$ and $y = \pi$. That is, the boundary conditions are

$$u(0, y, t) = u(\pi, y, t) = 0, \quad \frac{\partial}{\partial y} u(x, 0, t) = \frac{\partial}{\partial y} u(x, \pi, t) = 0$$

The initial position is $u(x, y, 0) = 0$ and initial velocity is $u_t(x, y, 0) = (6 - 5\cos 3y)\sin 4x$. There is an external force $Q(x, y, t) = 10e^{-t}\sin 2x$. Find an explicit solution for the associated PDE

$$u_{tt} = \nabla^2 u + Q.$$

Hint. Although the general solution is in terms of an infinite series with double indices, the particular solution should only consist of finite terms. In particular, there are only 3 modes being excited in this problem: $(2, 0), (4, 0)$ and $(4, 3)$.

Solution. The related S-L problem here is

$$\nabla^2 \phi + \lambda \phi = 0, \tag{1}$$

$$\text{with BC} \quad \phi(0, y) = \phi(\pi, y) = 0, \quad \frac{\partial}{\partial y}\phi(x, 0) = \frac{\partial}{\partial y}\phi(x, \pi) = 0$$

By using separation of variables $\phi(x, y) = X(x)Y(y)$, we can find the solutions (details skipped):

$$\lambda_{n,m} = n^2 + m^2, \quad \phi_{n,m}(x, y) = \sin nx \cos my, \quad n = 1, 2, 3, ..., \ m = 0, 1, 2, ...$$

Then, we use eigenfunction expansion to seek solutions in the form

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}(t)\phi_{n,m}(x, y)$$

By term-by-term differentiation,

$$u_{tt}(x, y, t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c''_{n,m}(t)\phi_{n,m}(x, y)$$

$$\nabla^2 u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}(t)\nabla^2\phi_{n,m}(x,y) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}(t)(-\lambda_{n,m}\phi_{n,m}(x,y)$$

The last equality above is due to the fact that $\phi_{n,m},\ \lambda_{n,m}$ solve the S-L problem (1). Next, apply eigenfunction expansion on the source term Q. Notice that the $\sin 2x$ term in Q corresponds to $\phi_{2,0}(x,y)$, and therefore

$$Q = \sum_n\sum_m q_{n,m}(t)\phi_{n,m}(x,y), \qquad \text{for } q_{2,0} = 10e^{-t} \text{ and } q_{n,m} = 0 \text{ otherwise.}$$

Therefore, by plugging all these series into the original PDE $u_{tt} = \nabla^2 u + Q$ and compare the corresponding coefficients of $\phi_{n,m}$, we find that

$$c''_{n,m}(t) + \lambda_{n,m}c_{n,m}(t) = \begin{cases} 10e^{-t}, & (n,m) = (2,0) \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

To find the initial conditions $c_{n,m}(0)$, $c'_{n,m}(0)$ for this family of ODEs, we apply the initial conditions.

$$u(x,y,0) = 0 = \sum_{n,m} c_{n,m}(0)\phi_{n,m}(x,y) \implies c_{n,m}(0) = 0, \tag{3}$$

$$u_t(x,y,0) = (6 - 5\cos 3y)\sin 4x = \sum_{n,m} c'_{n,m}(0)\phi_{n,m}(x,y) \implies$$

$$c'_{n,m}(0) = \begin{cases} 6, & (n,m) = (4,0) \\ -5, & (n,m) = (4,3) \\ 0, & \text{otherwise} \end{cases}. \tag{4}$$

Combining these initial conditions with equation (2), we immediately find that

$$c_{n,m}(y) = 0 \quad \text{for} \quad (n,m) \neq (2,0),\ (4,0),\ (4,3).$$

For $(n,m) = (2,0)$, equation (2) becomes an nonhomogeneous linear ODE

$$c''_{2,0}(t) + 4c_{2,0}(t) = 10e^{-t}$$

. By techniques learned from ODE courses, we "guess" a <u>particular</u> solution $c_p(t) = ke^{-t}$, plug it into the above equation and solve for k to arrive at $c_p(t) = 2e^{-t}$. Superposition c_p with the general solution to <u>homogeneous</u> equation $c''(t)+4c(t) = 0$ yields the general solution to equation (2) for $(n,m) = (2,0)$,

$$c_{2,0}(t) = 2e^{-t} + a\cos 2t + b\sin 2t.$$

The coefficients a, b are then determined from the initial conditions (3) and (4), that is $c_{2,0}(0) = 0 = 2 + a$ and $c'_{2,0}(0) = 0 = -2 + 2b \implies a = -2,\ b = 1$. Therefore,

$$c_{2,0}(t) = 2e^{-t} - 2\cos 2t + \sin 2t.$$

For $(n, m) = (4, 0)$, $(4, 3)$, the process is easier and we simply state that

$$c_{4,0}(t) = 1.5\cos 4t, \quad c_{4,3}(t) = -\cos 5t.$$

Therefore, the final solution is

$$u(x, y, t) = (2e^{-t} - 2\cos 2t + \sin 2t)\sin 2x + 1.5\cos 4t \sin 4x - \cos 5t \sin 4x \cos 3y$$

2. (**Heat equation with nonhomogeneous boundary conditions**.) Consider a 1D heat equation

$$u_t = k u_{xx}.$$

Suppose the left end is insulated and the right end is immersed in a media with cooling temperature. That is, prescribe the boundary conditions as

$$u_x(0, t) = 0, \qquad u(L, t) = a(e^{-t} - 1).$$

The initial temperature is $u(x, 0) = f(x)$. Find a solution formula to this problem. Also, use a computer program to obtain an approximation of the asymptotic temperature distribution as $t \to \infty$.

Solution. The associated S-L problem is

$$\phi''(x) + \lambda\phi(x) = 0,\ \phi'(0) = \phi(L) = 0.$$

(Notice the type of B.C. here is consistent with the original PDE.) Without further details, the solution family to the above problem is

$$\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2, \quad \phi_n(x) = \cos(\sqrt{\lambda_n}x), \quad n = 1, 2, 3...$$

By eigenfunction expansion, we

Step 1. Set up the eigenfunction series for the unknown

$$u(x, t) = \sum_{n=1}^{\infty} c_n(t)\phi_n(x), \tag{5}$$

where the variable coefficient c_n depends on t and is related to $u(x,t)$ by the coefficient formula

$$c_n(t) = \frac{\int_0^L u(x,t)\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx}. \tag{6}$$

Step 2. Expand the LHS of the PDE by applying term-by-term differentiation to (5)

$$u_t = \sum_{n=1}^{\infty} c_n'(t)\phi_n(x)$$

where, again, we can invole the cofficient formula to obtain

$$c_n'(t) = \frac{\int_0^L u_t(x,t)\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx}. \tag{7}$$

Step 3. Replace u_t in (7) with the RHS of the PDE,

$$c_n'(t) = \frac{\int_0^L u_{xx}\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx}$$

and continue with applying the Green's identity in 1D (that is $\int_0^L f''g - g''f\,dx = (f'g - g'f)|_0^L \implies \int_0^L f''g\,dx = \int_0^L g''f\,dx + (f'g - g'f)|_0^L$)

$$c_n'(t) = \frac{\int_0^L u\phi_n''(x)\,dx + (u_x\phi_n - u\phi_n')|_0^L}{\int_0^L \phi_n^2(x)\,dx}. \tag{8}$$

Evaluate the first term on the numerator using the definition of the S-L problem

$$\phi'' + \lambda\phi = 0 \implies \int_0^L u\phi_n''\,dx = -\lambda_n \int_0^L u\phi_n\,dx$$

and the second term using the BC for u and ϕ (note that we don't have to know the values for all the terms below since some terms automatically vanish due to a zero factor)

$$\begin{aligned}(u_x\phi_n - u\phi_n')|_0^L &= (u_x(L,t)\phi_n(L) - u(L,t)\phi_n'(L)) - (u_x(0,t)\phi_n(0) - u(0,t)\phi_n'(0)) \\ &= u(L,t)\phi_n'(L) = a(e^{-t}-1)\sqrt{\lambda_n}(-1)^{n-1}\end{aligned}$$

All in all, equation (8) is reduced to

$$c_n'(t) = \frac{-\lambda_n \int_0^L u\phi_n\,dx}{\int_0^L \phi_n^2\,dx} + \frac{a(e^{-t}-1)\sqrt{\lambda_n}(-1)^{n-1}}{\int_0^L \phi_n^2\,dx}$$

and by (6), the first term above is just $-\lambda_n c_n(t)$. The second term is $\frac{2}{L}a(e^{-t} - 1)\sqrt{\lambda_n}(-1)^{n-1}$.

Step 4. Solve the ODE derived from last step

$$c_n'(t) = -\lambda_n c_n(t) + \frac{2a}{L}(e^{-t} - 1)\sqrt{\lambda_n}(-1)^{n-1}$$

using integrating factor $e^{\lambda_n t}$. Multiply the above equation with the integrating factor and rearrange

$$\frac{d}{dt}(c_n(t)e^{\lambda_n t}) = \frac{2a}{L}(e^{(\lambda_n-1)t} - e^{\lambda_n t})\sqrt{\lambda_n}(-1)^{n-1}$$

$$\implies c_n(t)e^{\lambda_n t} - c_n(0) = \frac{2a}{L}(\frac{e^{(\lambda_n-1)t} - 1}{\lambda_n - 1} - \frac{e^{\lambda_n t} - 1}{\lambda_n})\sqrt{\lambda_n}(-1)^{n-1}$$

We can easily solve for $c_n(t)$ from the above equation.The initial condition $c_n(0)$ is derived from the IC for the PDE

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n(0)\phi_n(x) \implies c_n(0) = \frac{2}{L}\int_0^L f(x)\phi_n(x)\,dx.$$

To conclude, the solution to the PDE is $u(x,t) = \sum_{n=1}^{\infty} c_n(t)\cos\left(\frac{(2n-1)\pi x}{2L}\right)$ where

$$c_n(t) = e^{-\lambda_n t}\frac{2}{L}\int_0^L f(x)\phi_n(x)\,dx + \frac{2a}{L}\left(\frac{e^{-t} - e^{-\lambda_n}}{\lambda_n - 1} - \frac{1 - e^{-\lambda_n t}}{\lambda_n}\right)\sqrt{\lambda_n}(-1)^{n-1}$$

By letting $t \to +\infty$, we see that

$$\lim_{t\to\infty} c_n(t) = -\frac{2a}{L\lambda_n\sqrt{\lambda_n}(-1)^{n-1}} = -\frac{2a}{L\sqrt{\lambda_n}}(-1)^{n-1}.$$

So, formally, we derive that

$$\lim_{t\to\infty} u(x,t) = \sum_{n=1}^{\infty} -\frac{2a}{L\sqrt{\lambda_n}}(-1)^{n-1}\cos\left(\frac{(2n-1)\pi x}{2L}\right).$$

If it is properly plotted, one should see that $\lim_{t\to\infty} u(x,t) = -a$.

3. (**Green's function on a disk.**) Consider a quarter disk with radius A. Find a solution formula for $v(r,\theta)$ in the following Poisson's equation using eigenfunciton expansion and Green's function,

$$\nabla^2 v(r,\theta) = f(r,\theta),$$

$$|v(0,\theta)| < \infty, \quad v(A,\theta) = g(\theta), \text{ for } \theta \in (0,\pi/2),$$

$$v(r,0) = v(r,\pi/2) = 0, \text{ for } r \in [0,A].$$

(Note that if polar coordinates $d\theta dr$ are used for the integrals, add a weight function r. Otherwise, simply use $dxdy$.)

Solution. There are 3 major steps in solving a Poisson equation $L[u] = f$ on Ω and $u|_{\partial\Omega} = g$: first, set up the associated S-L problem $L[\phi] + \lambda\phi = 0$ and $\phi|_{\partial\Omega} = 0$, solve the S-L problem; secondly, obtain the Green's function using eigenfunction expansion (this step always leads to the same formula $G(x,y;x_0,y_0) = \sum_\lambda (-\phi_\lambda(x_0,y_0)\phi_\lambda(x,y)/\int_\Omega \phi_\lambda^2$ with notations suitably adapted); thirdly, use the Green's function together with the Green's identity to solve for the Poisson's equation (this step is always similar, too).

4. (**Green's function in a box.**) Consdier a 3D cube with side length π

$$\Omega = [0,\pi] \times [0,\pi] \times [0,\pi]$$

Find the Green's function $G(x,y,z;x_0,y_0,z_0)$ satisfying

$$\left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y_0^2} + \frac{\partial^2}{\partial z_0^2}\right) G(x,y,z;x_0,y_0,z_0) = \delta(x-x_0, y-y_0, z-z_0) \text{ for } (x_0,y_0,z_0) \in \Omega$$

$$G(x,y,z;x_0,y_0,z_0) = 0 \text{ for } (x_0,y_0,z_0) \in \partial\Omega$$

(Answer: $G(x,y,z;x_0,y_0,z_0) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty} \frac{-1}{n^2+m^2+l^2} \frac{\sin nx \sin my \sin lz \sin nx_0 \sin my_0 \sin lz_0}{(\pi/2)^3}$)

5. (**Fourier transform.**) Solve the following 1D heat equation for $u(x,t)$ with $x \in (-\infty,\infty)$, $t \in [0,\infty)$,

$$\frac{\partial}{\partial t}u(x,t) = k\frac{\partial^2}{\partial x^2}u(x,t) + e^{-t}g(x)$$

$$u(x,0) = f(x).$$

Express the solution in terms of Fourier transforms.