# Math 454 HW 5

Due: Nov 17 at noon

1. (**Multidimensional Sturm-Liouville problem.**) The motion of a vibrating membrane with damping can be described as

$$u_{tt} = c(x,y)^2 \nabla^2 u - k u_t,$$

$$\text{BC: } u(x,y,t) = 0 \text{ for } (x,y) \in \partial\Omega.$$

Here $k > 0$ is the damping constant.

(a) By using separation of variables $u(x,y,t) = \phi(x,y)G(t)$, find the associated Sturm-Liouville problem with proper boundary conditions. Find the ODE for $G(t)$ as well.

**Solution.** Replace $u$ with $\phi G$ in the PDE,

$$\phi(x,y)G''(t) = \left(c^2(x,y)\nabla^2\phi(x,y)\right) G(t) - k\phi(x,y)G'(t)$$

Divide it with $\phi G$ and rearrange

$$\frac{G'' + kG'}{G} = \frac{c^2\nabla^2\phi}{\phi} = -\lambda.$$

So the S-L problem is

$$\nabla^2\phi + \frac{\lambda}{c^2(x,y)}\phi = 0, \qquad \text{with} \qquad \phi|_{\partial\Omega} = 0.$$

The $G$ equation is

$$G'' + kG' + \lambda G = 0.$$

(b) Prove the self-adjointness of the operator obtained from part (a).

**Solution.** The related operator is $\mathcal{L}[u] = \nabla^2 u$. Pick any smooth testing function $u$, $v$ satisfying the boundary conditions $u|_\Omega = v|_\Omega = 0$, we apply the Green's identity to compute

$$\int_\Omega u\nabla^2 v - v\nabla^2 u \, dxdy = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\, dS = 0.$$

Therefore, $\mathcal{L}$ is self-adjoint w.r.t. the given BC.

(c) State the orthogonality condition implied by part (b). Note the weight function $\sigma \neq 1$.

**Solution.** For any two different eigenvalues $\lambda_n \neq \lambda_m$, the associated eigenfunctions satisfy

$$\int_\Omega \phi_n \phi_m \frac{1}{c^2(x,y)} \, dxdy = 0.$$

(d) Assume the solutions to the Sturm-Liouville problem from part (a) are given as $\lambda_{n,m}$, $\phi_{n,m}$. Use this information to find the general solution for the original PDE in terms of infinite series.

**Solution.** We need to solve for $G(t)$. For any fixed $n$, the $G$ equation is

$$G''(t) + kG'(t) + \lambda_n G(t) = 0.$$

The characteristic equation for this ODE is $r^2 + kr + \lambda_n = 0$ for which the solution is

$$r = \frac{-k \pm \sqrt{k^2 - 4\lambda_n}}{2}.$$

Thus,

$$G_n(t) = a \exp(\frac{-k + \sqrt{k^2 - 4\lambda_n}}{2} t) + b \exp(\frac{-k - \sqrt{k^2 - 4\lambda_n}}{2} t) \quad \text{if} \quad k^2 \neq 4\lambda_n$$

$$\text{and} \quad G_n(t) = a \exp(-\frac{k}{2} t) + bt \exp(-\frac{k}{2} t) \quad \text{if} \quad k^2 = 4\lambda_n.$$

So the general solution is

$$u(x,y,t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \phi_{n,m} \left( a_{n,m} \exp(\frac{-k + \sqrt{k^2 - 4\lambda_{n,m}}}{2} t) + b_{n,m} \exp(\frac{-k - \sqrt{k^2 - 4\lambda_{n,m}}}{2} t) \right)$$

$$\text{if} \quad k^2 \neq \quad \text{any} \quad \lambda_{n,m}$$

$$u(x,y,t) = \sum_{(n,m) \neq (n^*,m^*)} \phi_{n,m} \left( a_{n,m} \exp(\frac{-k + \sqrt{k^2 - 4\lambda_{n,m}}}{2} t) + b_{n,m} \exp(\frac{-k - \sqrt{k^2 - 4\lambda_{n,m}}}{2} t) \right)$$

$$+ \phi_{n^*,m^*} \left( a_{n^*,m^*} \exp(-\frac{k}{2} t) + b_{n^*,m^*} t \exp(-\frac{k}{2} t) \right) \quad \text{if} \quad k^2 = \quad \text{some} \quad 4\lambda_{n^*,m^*}$$

(e) If the initial conditions are

$$u(x,y,0) = f(x,y), \quad \frac{\partial}{\partial t}u(x,y,0) = g(x,y),$$

find the formula for the coefficients of the series in part (d).

2. (**Heat equation in a 3D box**). Solve the following PDE for $u(x,y,z,t)$ on spatial domain $\Omega = \{(x,y,z)|x \in [0,L_1], y \in [0,L_2], z \in [0,L_3]\}$. The boundary condition is of type 2 (insulation). Use a graph if necessary.

$$\frac{\partial}{\partial t}u(x,y,z,t) = k\nabla^2 u(x,y,z,t), \quad \text{for} \quad (x,y,z) \in \Omega,$$

$$\frac{\partial}{\partial n}u(x,y,z,t) = 0, \quad \text{for} \quad (x,y,z) \in \partial\Omega,$$
$$u(x,y,z,0) = f(x,y,z), \quad \text{for} \quad (x,y,z) \in \Omega.$$

**Solution.** Set $u(x,y,z,t) = X(x)Y(y)Z(z)T(t)$ in the PDE

$$XYZT' = k\left(X''YZT + XY''ZT + XYZ''T\right)$$

that is

$$\frac{T'}{kT} = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -\lambda \text{ (a constant)}$$

So ODE for $T$

$$T' + k\lambda T = 0, \tag{1}$$

and a PDE for $X,Y,Z$,

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = -\lambda. \tag{2}$$

Upon rearrangement

$$\frac{Y''}{Y} + \frac{Z''}{Z} + \lambda = -\frac{X''}{X} = \alpha \text{ another constant.}$$

So ODE for $X$,

$$X'' + \alpha X = 0, \text{ with BC } X'(0) = X'(L_1) = 0. \tag{3}$$

Likewise, we obtain ODEs for $Y, Z$

$$Y'' + \beta Y = 0, \text{ with BC } Y'(0) = Y'(L_2) = 0. \tag{4}$$

$$Z'' + \gamma Z = 0, \text{ with BC } Z'(0) = Z'(L_3) = 0. \tag{5}$$

And, the eigenvalue $\lambda$ for (2) equals

$$\lambda = \alpha + \beta + \gamma.$$

Solve (3), (4), (5) respectively

$$\alpha_n = \left(\frac{n\pi}{L_1}\right)^2, \ X_n = \cos(\frac{n\pi x}{L_1}), \ n = 0, 1, 2, ...$$

$$\beta_m = \left(\frac{m\pi}{L_2}\right)^2, \ Y_m = \cos(\frac{m\pi y}{L_2}), \ m = 0, 1, 2, ...$$

$$\gamma_k = \left(\frac{k\pi}{L_3}\right)^2, \ Z_k = \cos(\frac{k\pi z}{L_3}), \ k = 0, 1, 2, ...$$

Solve (1) with $\lambda_{mnk} = \alpha_n + \beta_m + \gamma_k = \left(\frac{n\pi}{L_1}\right)^2 + \left(\frac{m\pi}{L_2}\right)^2 + \left(\frac{k\pi}{L_3}\right)^2$,

$$T(t) = T(0)e^{-k\lambda_{nmk}t}.$$

Therefore, the solution formula is

$$u = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} c_{nmk}e^{-k\lambda_{nmk}t}\phi_{nmk}(x, y, z),$$

where

$$\phi_{nmk} = \cos(\frac{n\pi x}{L_1})\cos(\frac{m\pi y}{L_2})\cos(\frac{k\pi z}{L_3})$$

and coefficents

$$c_{nmk} = \frac{\int_\Omega f(x, y, z)\phi_{nmk}(x, y, z)}{\int_\Omega |\phi_{nmk}(x, y, z)|^2}$$

3. (**Rayleigh Quotient and positivity of eigenvalues.**) Consider the 2D eigenvalue problem

$$\nabla \cdot (p(x, y)\nabla\phi(x, y)) + q(x, y)\phi(x, y) + \lambda\sigma(x, y)\phi(x, y) = 0,$$

subject to boundary condition

$$a\phi(x, y) + b\frac{\partial}{\partial n}\phi(x, y) = 0, \quad \text{for} \quad (x, y) \in \partial\Omega.$$

Here, $p(x, y) > 0$, $q(x, y) \leq 0$, $\sigma(x, y) > 0$ and $ab > 0$. Use the Rayleigh quotient to show that $\lambda \geq 0$.

**Solution.** By the Rayleigh quotient in multi-dimension

$$\lambda = \frac{-\int_{\partial\Omega} p\phi \frac{\partial \phi}{\partial n}\, dS - \int_\Omega q(x,y)|\phi(x,y)|^2\, dxdy + \int_\Omega p(x,y)|\nabla\phi(x,y)|^2}{\int_\Omega \phi^2(x,y)\sigma(x,y)\, dxdy}. \tag{6}$$

Each term on the RHS is nonnegative. In particular,

$$-\int_{\partial\Omega} p\phi\frac{\partial\phi}{\partial n}\, dS = \int_{\partial\Omega}\left(\frac{a}{b}\right) p\phi^2\, dS \geq 0$$

due to the BC $a\phi + b\frac{\partial\phi}{\partial n} = 0$ and the fact $ab > 0$. Therefore, we show that$\lambda \geq 0$.

4. (**Bessel functions**.) Consider one sixth of a 2-D disk with radius 1, $\Omega = \{(r,\theta)|0 \leq r \leq 1,\ 0 \leq \theta \leq \pi/3\}$. Use a graph if necessary.

   (a) With Bessel functions, find all eigenvalues and associated eigenfunctions for the following Sturm-Liouville problem.

$$\nabla^2\phi(r,\theta) + \lambda\phi(r,\theta) = 0, \quad \text{for} \quad (r,\theta) \in \Omega$$

$$\phi(1,\theta) = 0, \quad \text{for} \quad \theta \in [0,\pi/3],$$

$$\phi(r,0) = \phi(r,\pi/3) = 0 \quad \text{for} \quad r \in [0,1],$$

$$|\phi(0,\theta)| < \infty.$$

**Solution.** The solution follows Handout 2, Step 1-5 with the following changes,

- Equation (3) becomes

$$\Theta'' + \mu\Theta = 0 \text{ with B.C. of type 1 } \Theta(0) = \Theta(\pi/3) = 0. \tag{7}$$

- Equation (4) becomes

$$\mu_m = (3m)^2, \Theta_m(\theta) = \sin(3m\theta), \text{ for } m = 1,2,3,... \tag{8}$$

- Every occurrence of $m$ in Step 3 is replaced with $3m$.
- Every $J_m$ in Step 4 is replaced with $J_{3m}$ and $A$ is replaced with 1. Especially the last sentence becomes: By labeling the zeros of $J_k(z)$ as $z_{k,1}, z_{k,2}, z_{k,3}, ...$, we have

$$\sqrt{\lambda_{m,n}} = z_{3m,n} \implies \lambda_{m,n} = (z_{3m,n})^2 \text{ and } R_{m,n}(r) = J_{3m}(\sqrt{\lambda_{m,n}}\, r). \tag{9}$$

– Step 5 becomes: The eigenvalues $\lambda$ and associate eigenfunctions $\phi(r,\theta) = R(r)\Theta(\theta)$ to the original S-L problem is

$$\lambda_{m,n} = (z_{3m,n})^2, \quad \phi_{m,n} = \sin(3m\theta) J_{3m}(\sqrt{\lambda_{m,n}}\, r),$$

$$\text{for } m = 1,2,3..., \ n = 1,2,3,...$$

(b) Use part (a) to find a solution formula for $u(r,\theta,t)$ in the following heat equation.

$$\begin{gathered} u_t = k\nabla^2 u, \\ u(1,\theta,t) = 0, \quad \text{for} \quad \theta \in [0,\pi/3], \\ u(r,0,t) = u(r,\pi/3,t) = 0 \quad \text{for} \quad r \in [0,1], \\ u(r,\theta,0) = f(r,\theta), \quad \text{for} \quad (r,\theta) \in \Omega. \end{gathered}$$

**Solution.** Following the usual steps, we have

$$u(r,\theta,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{m,n}\phi_{m,n} e^{-k\lambda_{m,n}t}$$

where the coefficient $c_{m,n}$ is given by the fomula

$$c_{m,n} = \frac{\int_{\theta=0}^{\theta=\pi/3}\int_{r=0}^{r=1} f(r,\theta)\phi_{m,n}(r,\theta) r\, dr d\theta}{\int_{\theta=0}^{\theta=\pi/3}\int_{r=0}^{r=1} \phi_{m,n}^2(r,\theta) r\, dr d\theta}$$

(c) Consider a perturbed version of part (a),

$$\begin{gathered} \nabla^2\phi(r,\theta) - (2+\sin r)\phi(r,\theta) + \lambda\phi(r,\theta) = 0, \\ \phi(1,\theta) = 0, \quad \text{for} \quad \theta \in [0,\pi/3], \\ \phi(r,0) = \phi(r,\pi/3) = 0 \quad \text{for} \quad r \in [0,1], \\ |\phi(0,\theta)| < \infty. \end{gathered} \tag{10}$$

Derive the Rayleigh Quotient here and compare it with the Rayleigh Quotient associated with part (a). Then, use the solution of part (a) to find a suitable range for the smallest eigenvalue $\lambda_{min}$ here in part (c). Google "Bessel function zeros" for numerical values needed. Note that $\lambda_{min} > 0$ is not accurate enough.

**Solution.** Multiply the first line of (10) with $\phi$ and integrate over $\Omega$. By the Green's identities, the first term yields

$$\int_\Omega \phi\nabla^2\phi dxdy = \int_{\partial\Omega} \phi\frac{\partial\phi}{\partial n}\, dS - \int_\Omega |\nabla\phi|^2\, dxdy.$$

Applying the BC $\phi|_{\partial\Omega} \equiv 0$, we have

$$\int_\Omega \phi\nabla^2\phi dxdy = -\int_\Omega |\nabla\phi|^2 \, dxdy = -\int_{\theta=0}^{\theta=\pi/3}\int_{r=0}^{r=1} |\nabla\phi(r,\theta)|^2 r \, drd\theta.$$

Notice here we can either use the Cartesian form or the polar form of the integrals over $\Omega$. In the following, we use the Cartesian form.

Thus, the Rayleigh Quotient associated with (10) is

$$R_{4c}[\phi] = \frac{\int_\Omega |\nabla\phi|^2 \, dxdy + \int_\Omega (2+\sin\sqrt{x^2+y^2})\phi^2 \, dxdy}{\int_\Omega \phi^2 \, dxdy}. \tag{11}$$

Since $0 \le \sin\sqrt{x^2+y^2} \le \sin(1)$ on $\Omega$, we have

$$\int_\Omega 2\phi^2 \, dxdy \le \int_\Omega (2+\sin\sqrt{x^2+y^2})\phi^2 \, dxdy \le \int_\Omega (2+\sin(1))\phi^2 \, dxdy$$

and therefore by (11), we have

$$2 + \frac{\int_\Omega |\nabla\phi|^2 \, dxdy}{\int_\Omega \phi^2 \, dxdy} \le R_{4c}[\phi] \le 2+\sin(1) + \frac{\int_\Omega |\nabla\phi|^2 \, dxdy}{\int_\Omega \phi^2 \, dxdy}.$$

On the other hand, it is easy to derive that the Rayleigh Quotient associated with part (a) is

$$R_{4a}[\phi] = \frac{\int_\Omega |\nabla\phi|^2 \, dxdy}{\int_\Omega \phi^2 \, dxdy}.$$

Therefore, we see that these two Rayleigh Quotients are related as

$$2 + R_{4a}[\phi] \le R_{4c}[\phi] \le 2 + \sin(1) + R_{4a}[\phi].$$

Taking the minumum of the above inequalities for all $\phi$ satisfying the BC $\phi|_{\partial\Omega} \equiv 0$, we have

$$2 + \min_\phi R_{4a}[\phi] \le \min_\phi R_{4c}[\phi] \le 2 + \sin(1) + \min_\phi R_{4a}[\phi].$$

Since the minimum of Rayleigh Quotient of a given S-L problem always equals the smallest eigenvalue thereof, we see that

$$2 + \lambda_{1,1} \le \lambda_{min} \le 2 + \sin(1) + \lambda_{1,1}$$

where $\lambda_{1,1}$ is given in (9) as $z_{3,1}^2$, i.e. the first zero of $J_3(z)$. Looking it up online, we find $z_{3,1} = 6.3802$ and therefore

$$2 + 6.3802^2 \le \lambda_{min} \le 2 + \sin(1) + 6.3802^2.$$