Math 454 HW 5

Due: Nov 17 at noon

1. (**Multidimensional Sturm-Liouville problem.**) The motion of a vibrating membrane with damping can be described as

$$u_{tt} = c(x,y)^2 \nabla^2 u - k u_t,$$

$$\text{BC: } u(x,y,t) = 0 \text{ for } (x,y) \in \partial\Omega.$$

Here $k > 0$ is the damping constant.

(a) By using separation of variables $u(x,y,t) = \phi(x,y)G(t)$, find the associated Sturm-Liouville problem with proper boundary conditions. Find the ODE for $G(t)$ as well.

Solution. Replace u with ϕG in the PDE,

$$\phi(x,y)G''(t) = \left(c^2(x,y)\nabla^2\phi(x,y)\right) G(t) - k\phi(x,y)G'(t)$$

Divide it with ϕG and rearrange

$$\frac{G'' + kG'}{G} = \frac{c^2\nabla^2\phi}{\phi} = -\lambda.$$

So the S-L problem is

$$\nabla^2\phi + \frac{\lambda}{c^2(x,y)}\phi = 0, \qquad \text{with} \qquad \phi|_{\partial\Omega} = 0.$$

The G equation is

$$G'' + kG' + \lambda G = 0.$$

(b) Prove the self-adjointness of the operator obtained from part (a).

Solution. The related operator is $\mathcal{L}[u] = \nabla^2 u$. Pick any smooth testing function u, v satisfying the boundary conditions $u|_\Omega = v|_\Omega = 0$, we apply the Green's identity to compute

$$\int_\Omega u\nabla^2 v - v\nabla^2 u \, dxdy = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\, dS = 0.$$

Therefore, $\mathcal{L}$ is self-adjoint w.r.t. the given BC.

(c) State the orthogonality condition implied by part (b). Note the weight function $\sigma \neq 1$.

Solution. For any two different eigenvalues $\lambda_n \neq \lambda_m$, the associated eigenfunctions satisfy

$$\int_\Omega \phi_n \phi_m \frac{1}{c^2(x,y)}\, dxdy = 0.$$

(d) Assume the solutions to the Sturm-Liouville problem from part (a) are given as $\lambda_{n,m}$, $\phi_{n,m}$. Use this information to find the general solution for the original PDE in terms of infinite series.

Solution. We need to solve for $G(t)$. For any fixed n, the G equation is

$$G''(t) + kG'(t) + \lambda_n G(t) = 0.$$

The characteristic equation for this ODE is $r^2 + kr + \lambda_n = 0$ for which the solution is

$$r = \frac{-k \pm \sqrt{k^2 - 4\lambda_n}}{2}.$$

Thus,

$$G_n(t) = a\exp(\frac{-k + \sqrt{k^2 - 4\lambda_n}}{2}t) + b\exp(\frac{-k - \sqrt{k^2 - 4\lambda_n}}{2}t) \quad \text{if} \quad k^2 \neq 4\lambda_n$$

$$\text{and} \quad G_n(t) = a\exp(-\frac{k}{2}t) + bt\exp(-\frac{k}{2}t) \quad \text{if} \quad k^2 = 4\lambda_n.$$

So the general solution is

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \phi_{n,m}\left(a_{n,m}\exp(\frac{-k + \sqrt{k^2 - 4\lambda_{n,m}}}{2}t) + b_{n,m}\exp(\frac{-k - \sqrt{k^2 - 4\lambda_{n,m}}}{2}t)\right)$$

$$\text{if} \quad k^2 \neq \quad \text{any} \quad \lambda_{n,m}$$

$$u(x,y,t) = \sum_{(n,m)\neq(n^*,m^*)} \phi_{n,m}\left(a_{n,m}\exp(\frac{-k + \sqrt{k^2 - 4\lambda_{n,m}}}{2}t) + b_{n,m}\exp(\frac{-k - \sqrt{k^2 - 4\lambda_{n,m}}}{2}t)\right)$$

$$+\phi_{n^*,m^*}\left(a_{n^*,m^*}\exp(-\frac{k}{2}t) + b_{n^*,m^*}t\exp(-\frac{k}{2}t)\right) \quad \text{if} \quad k^2 = \quad \text{some} \quad 4\lambda_{n^*,m^*}$$

(e) If the initial conditions are

$$u(x,y,0) = f(x,y), \quad \frac{\partial}{\partial t}u(x,y,0) = g(x,y),$$

find the formula for the coefficients of the series in part (d).

2. (**Heat equation in a 3D box**). Solve the following PDE for $u(x,y,z,t)$ on spatial domain $\Omega = \{(x,y,z)|x \in [0,L_1], y \in [0,L_2], z \in [0,L_3]\}$. The boundary condition is of type 2 (insulation). Use a graph if necessary.

$$\frac{\partial}{\partial t}u(x,y,z,t) = k\nabla^2 u(x,y,z,t), \quad \text{for} \quad (x,y,z) \in \Omega,$$

$$\frac{\partial}{\partial n}u(x,y,z,t) = 0, \quad \text{for} \quad (x,y,z) \in \partial\Omega,$$
$$u(x,y,z,0) = f(x,y,z), \quad \text{for} \quad (x,y,z) \in \Omega.$$

3. (**Rayleigh Quotient and positivity of eigenvalues.**) Consider the 2D eigenvalue problem

$$\nabla \cdot (p(x,y)\nabla\phi(x,y)) + q(x,y)\phi(x,y) + \lambda\sigma(x,y)\phi(x,y) = 0,$$

subject to boundary condition

$$a\phi(x,y) + b\frac{\partial}{\partial n}\phi(x,y) = 0, \quad \text{for} \quad (x,y) \in \partial\Omega.$$

Here, $p(x,y) > 0$, $q(x,y) \leq 0$, $\sigma(x,y) > 0$ and $ab > 0$. Use the Rayleigh quotient to show that $\lambda \geq 0$.

4. (**Bessel functions**.) Consider one sixth of a 2-D disk with radius 1, $\Omega = \{(r,\theta)|0 \leq r \leq 1,\ 0 \leq \theta \leq \pi/3\}$. Use a graph if necessary.

(a) With Bessel functions, find all eigenvalues and associated eigenfunctions for the following Sturm-Liouville problem.

$$\nabla^2\phi(r,\theta) + \lambda\phi(r,\theta) = 0, \quad \text{for} \quad (r,\theta) \in \Omega$$

$$\phi(1,\theta) = 0, \quad \text{for} \quad \theta \in [0,\pi/3],$$

$$\phi(r,0) = \phi(r,\pi/3) = 0 \quad \text{for} \quad r \in [0,1],$$

$$|\phi(0,\theta)| < \infty.$$

(b) Use part (a) to find a solution formula for $u(r, \theta, t)$ in the following heat equation.

$$
\begin{gathered}
u_t = k\nabla^2 u, \\
u(1, \theta, t) = 0, \quad \text{for} \quad \theta \in [0, \pi/3], \\
u(r, 0, t) = u(r, \pi/3, t) = 0 \quad \text{for} \quad r \in [0, 1], \\
u(r, \theta, 0) = f(r, \theta), \quad \text{for} \quad (r, \theta) \in \Omega.
\end{gathered}
$$

(c) Consider a perturbed version of part (a),

$$
\begin{gathered}
\nabla^2\phi(r, \theta) - (2 + \sin r)\phi(r, \theta) + \lambda\phi(r, \theta) = 0, \\
\phi(1, \theta) = 0, \quad \text{for} \quad \theta \in [0, \pi/3], \\
\phi(r, 0) = \phi(r, \pi/3) = 0 \quad \text{for} \quad r \in [0, 1], \\
|\phi(0, \theta)| < \infty.
\end{gathered}
$$

Derive the Rayleigh Quotient here and compare it with the Rayleigh Quotient associated with part (a). Then, use the solution of part (a) to find a suitable range for the smallest eigenvalue λ_{min} here in part (c). Google "Bessel function zeros" for numerical values needed. Note that $\lambda_{min} > 0$ is not accurate enough.