Math 454 HW 4

Due: Nov 3 at noon

1. (**Mixed boundary conditions**) Consider the boundary value problem

$$\phi'' + \lambda\phi = 0$$
$$\phi(0) - \phi'(0) = 0, \quad \phi(2) + \phi'(2) = 0.$$

   (a) Using the Rayleigh quotient, show that $\lambda \geq 0$.

   (b) Show that $\lambda > 0$ by proving that $\lambda = 0$ is impossible.

   (b) Show that, any eigenvalue λ satisfies,

$$\tan 2\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1}.$$

   Use Matlab/Maple/graphic calculator to plot both sides of the equation and graphically determine the smallest 3 eigenvalues.

2. (Approximation of eigenvalues) Use the Rayleigh quotient to obtain an approximate upper bound for the lowest eigenvalue of

$$\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0$$
$$\phi(x) = 0, \quad \frac{d\phi}{dx}(1) + 2\phi(1) = 0.$$

   Try at least two testing functions of the form $x^2 + bx + c$ and of the form $ax + d + \sin(\pi x)$. Note that the boundary conditions have to be always satisfied (so functions like $\sin(\pi x)$ won't do the job).

3. **Nonuniform vibrating string.** The 1D wave equation on a nonuniform string can be written as

$$u_{tt} = c^2(x)u_{xx}$$

   with

$$\text{BC: } u(0,t) = u(L,t) = 0,$$
$$\text{IC: } u(x,0) = f(x),\ u_t(x,0) = g(x).$$

   (a) By separation of variables, derive the associated Sturm-Liouvill problem.

(b) Show the self-adjointness of the Sturm-Liouville problem you obtain in (a).

(c) Assume the Sturm-Liouville problem has eigenvalues λ_n and associated eigenfunction $\phi_n(x)$. Find a solution formula for $u(x,t)$. Include the initial condition in your formula. (Note: the weight function $\sigma(x) = \dfrac{1}{c^2(x)}$.)

4. (**Integration by parts**) Physical intuition tells us that if an object has fixed temperature on the boundary and contains no thermal source/sink inside, then by heat diffusion the overall temperature should tend to an equilibrium. Let's us study this for a 1D thin rod with boundary condition of type 1,

$$\frac{\partial}{\partial t}u(x,t) = k\frac{\partial^2}{\partial x^2}u(x,t),$$

$$u(0,t) = u(L,t) = 0,$$

$$u(x,0) = f(x).$$

Consider the function that measures how far away the over temperature is from zero (notice the square here),

$$V(t) = \int_0^L u^2(x,t)\,dx.$$

Prove that, as long as $u(x,t)$ is not a constant, $V(t)$ always decreases with time, that is,

$$\frac{d}{dt}V(t) < 0 \qquad \text{unless} \qquad u(x,t) \equiv constant.$$

Hint: plug in the definition of $V(t)$, use the PDE to replace u_t with x derivatives and then apply integration by parts.