Math 454 HW 4

Due: Nov 3 at noon

1. (**Mixed boundary conditions**) Consider the boundary value problem

$$\phi'' + \lambda\phi = 0$$
$$\phi(0) - \phi'(0) = 0, \quad \phi(2) + \phi'(2) = 0.$$

(a) Using the Rayleigh quotient, show that $\lambda \geq 0$.

Solution. This is a 2nd order S-L problem with $p = 1$, $q = 0$, $\sigma = 1$ so the related Rayleigh quotient is

$$R[\phi] = \frac{-\phi(x)\phi'(x)|_{x=0}^{x=2} + \int_0^2 (\phi'(x))^2\, dx}{\int_0^2 \phi^2(x)\, dx}.$$

Here, both integrals have integrands in the form of square which are nonnegative. The boundary term is also nonnegative since, by the given BC in the problem, we can replace $\phi'(0)$ with $\phi(0)$ and replace $\phi'(2)$ with $-\phi(2)$,

$$-\phi(x)\phi'(x)|_{x=0}^{x=2} = -\phi(2)\phi'(2) + \phi(0)\phi'(0) = \phi^2(2) + \phi^2(0) \geq 0.$$

(b) Show that $\lambda > 0$ by proving that $\lambda = 0$ is impossible.

Solution. Prove by contradiction. Assume $\lambda = 0$. Then, in order for the Rayleigh quotient to be zero, each nonnegative term in the numerator has to be exactly zero

$$-\phi(x)\phi'(x)|_{x=0}^{x=2} = \phi^2(2) + \phi^2(0) = 0 \implies \phi(2) = \phi(0) = 0$$
$$\int_0^2 (\phi'(x))^2\, dx = 0 \implies \phi'(x) \equiv 0 \implies \phi(x) \equiv constant$$

Combine the above equations and we obtain $\phi(x) \equiv 0$. Trivial solution! Thus, assuming $\lambda = 0$ leads to contradiction and therefore

$$\lambda > 0.$$

(c) Show that, any eigenvalue λ satisfies,

$$\tan 2\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1}.$$

Use Matlab/Maple/graphic calculator to plot both sides of the equation and graphically determine the smallest 3 eigenvalues.

Solution. Since we know from part (b) that $\lambda > 0$, the general solution of the S-L problem must be of trigonometric forms

$$\phi(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

It's derivative is

$$\phi'(x) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x).$$

Apply the BC at $x = 0$,

$$\phi(0) - \phi'(0) = 0 = c_1 - c_2\sqrt{\lambda} \implies c_1 = c_2\sqrt{\lambda} \quad ... \quad (1)$$

Apply the BC at $x = 2$,

$$\phi(2)+\phi'(2) = 0 = c_1 \cos(\sqrt{\lambda}\,2)+c_2 \sin(\sqrt{\lambda}\,2)-c_1\sqrt{\lambda}\sin(\sqrt{\lambda}\,2)+c_2\sqrt{\lambda}\cos(\sqrt{\lambda}\,2) \quad ... \quad (2)$$

Plug (1) into (2) and factor out c_2,

$$0 = c_2\left(\sqrt{\lambda}\cos(\sqrt{\lambda}\,2) + \sin(\sqrt{\lambda}\,2) - \sqrt{\lambda}\sqrt{\lambda}\sin(\sqrt{\lambda}\,2) + \sqrt{\lambda}\cos(\sqrt{\lambda}\,2)\right)$$

For nontriviality, $c_2 \neq 0$. Therefore,

$$0 = \sqrt{\lambda}\cos(\sqrt{\lambda}\,2) + \sin(\sqrt{\lambda}\,2) - \sqrt{\lambda}\sqrt{\lambda}\sin(\sqrt{\lambda}\,2) + \sqrt{\lambda}\cos(\sqrt{\lambda}\,2)$$

Rearrange the terms with sine on one side and those with cosine on the other side

$$\sin(\sqrt{\lambda}\,2)(\lambda - 1) = 2\cos(\sqrt{\lambda}\,2)\sqrt{\lambda}$$

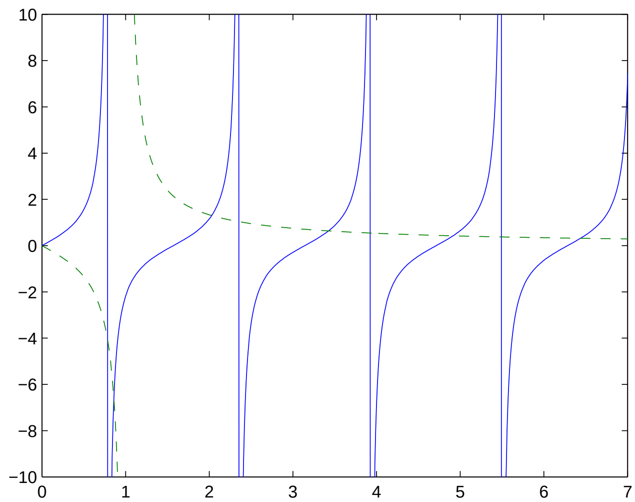
which readily implies

$$\tan(2\sqrt{\lambda}) = \frac{2\sqrt{\lambda}}{\lambda - 1}$$

The following graph is generated from Matlab command

```
x=0:0.01:3*pi; y=tan(2*x); z=2*x./(x.*x-1); plot(x,y,'-',x,z,'--');axis([0,7,-10,10]);
```

showing two plots, solid line for $\tan(2\sqrt{\lambda})$ vs. $\sqrt{\lambda}$ and dashed line for $\frac{2\sqrt{\lambda}}{\lambda-1}$ vs. $\sqrt{\lambda}$. Note that the horizontal axis stands for $\sqrt{\lambda}$, not λ. Also note that the straight vertical lines only indicate the asympotitcs of $\tan(2\sqrt{\lambda})$ and should NOT be considered as part of the actual plot.



Since $\sqrt{\lambda} > 0$, the intersection in the lower half plane should be discarded and the next 3 intersections corresponds to

$$\sqrt{\lambda} \approx 1.9, 3.3, 4.9 \implies$$

$$\lambda_1 \approx 1.9^2, \ \lambda_2 \approx 3.3^2, \ \lambda_3 \approx 4.9^2.$$

2. (Approximation of eigenvalues) Use the Rayleigh quotient to obtain an approximate upper bound for the lowest eigenvalue of

$$\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0$$
$$\phi(0) = 0, \quad \frac{d\phi}{dx}(1) + 2\phi(1) = 0.$$

Try at least two testing functions of the form $x^2 + bx + c$ and of the form $ax + d + \sin(\pi x)$. Note that the boundary conditions have to be always satisfied (so functions like $\sin(\pi x)$ won't do the job).

Solution. A suitable testing needs to satisfy the BC but not necessarily the equation itself. For the form $\phi(x) = x^2 + bx + c$, apply the BC

$$\phi(0) = 0 = c \implies \phi(x) = x^2 + bx,$$

$$\phi'(x) = 2x + b, \quad \phi'(1) + 2\phi(1) = 0 = 2 + b + 2(1 + b) \implies b = -4/3.$$

So

$$\phi(x) = x^2 - 4x/3 \quad \text{and} \quad \phi'(x) = 2x - 4/3.$$

The associated Rayleigh quotient is

$$\begin{aligned}
R[\phi(x)] =& \frac{-\phi(x)\phi'(x)|_0^1 + \int_0^1 (\phi'(x))^2\,dx + \int_0^1 x^2\phi^2(x)\,dx}{\int_0^1 \phi^2(x)\,dx} \\
=& \frac{2/9 + (4x^3/3 + 16x/9 - 8x^2/3)|_0^1 + (x^7/7 + 16x^5/45 - 4x^6/9)|_0^1}{(x^5/5 + 16x^3/27 - 2x^4/3)|_0^1} \\
=& \frac{0.720635}{0.125926} = 5.72269
\end{aligned}$$

For the form $\phi(x) = ax + d + \sin(\pi x)$, we have

$$\phi(0) = 0 = d \implies \phi(x) = ax + \sin(\pi x),$$

$$\phi'(x) = a + \pi\cos(\pi x), \quad \phi'(1) + 2\phi(1) = 0 = a - \pi + 2(a + 0) \implies a = \pi/3.$$

So

$$\phi(x) = \pi x/3 + \sin(\pi x), \quad \phi'(x) = \pi/3 + \pi\cos(\pi x).$$

The associated Rayleigh quotient is

$$\begin{aligned}
& R[\phi(x)] \\
=& \frac{-\phi(x)\phi'(x)|_0^1 + \int_0^1 (\phi'(x))^2\,dx + \int_0^1 x^2\phi^2(x)\,dx}{\int_0^1 \phi^2(x)\,dx} \\
=& \frac{\frac{2\pi^2}{9} + \int_0^1 \frac{\pi^2}{9} + \frac{\pi^2(1+\cos(2\pi x))}{2} + \frac{2\pi^2\cos(\pi x)}{3}\,dx + \int_0^1 \frac{\pi^2 x^4}{9} + \frac{x^2(1-\cos(2\pi x))}{2} + \frac{2\pi x^3\sin(\pi x)}{3}\,dx}{\int_0^1 \pi^2x^2/9 + (1-\cos(2\pi x))/2 + 2\pi x\sin(\pi x)/3}
\end{aligned}$$

Here, we used trig identities $\cos^2\theta = (1 + \cos 2\theta)/2$ and $\sin^2\theta = (1 - \cos 2\theta)/2$ Using integration by parts repetitively,

$$\int x\sin(\pi x)\,dx = -\frac{1}{\pi}x\cos(\pi x) + \frac{1}{\pi}\int \cos(\pi x)\,dx = -\frac{1}{\pi}x\cos(\pi x) + \frac{1}{\pi^2}\sin(\pi x) + C \; ... \; (3)$$

Therefore,

$$\int_0^1 x\sin(\pi x)\,dx = \frac{1}{\pi}$$

$$\begin{aligned}
\int x^2\cos(\pi x)\,dx =& \frac{1}{\pi}x^2\sin(\pi x) - \frac{1}{\pi}\int 2x\sin(\pi x)\,dx \quad \text{(int. by parts)} \\
=& \frac{1}{\pi}x^2\sin(\pi x) - \frac{2}{\pi}\left(-\frac{1}{\pi}x\cos(\pi x) + \frac{1}{\pi^2}\sin(\pi x)\right) + C \; ... \text{ by (3).}
\end{aligned}$$

Therefore,using substitution, we have

$$\begin{aligned}
&\int_0^1 x^2\cos(2\pi x)\,dx\\
=&\int_0^2 \left(\frac{x}{2}\right)^2\cos(\pi x)\,dx\\
=&\frac{1}{4}\left(\frac{1}{\pi}x^2\sin(\pi x)-\frac{2}{\pi}(-\frac{1}{\pi}x\cos(\pi x)+\frac{1}{\pi^2}\sin(\pi x))\right)\Bigg|_0^2\\
=&\frac{1}{2\pi^2}
\end{aligned}$$

And,

$$\begin{aligned}
\int x^3\sin(\pi x)\,dx &= -\frac{1}{\pi}x^3\cos(\pi x)+\frac{1}{\pi}\int 3x^2\cos(\pi x)\,dx \quad \text{(int. by parts)}\\
&= -\frac{1}{\pi}x^3\cos(\pi x)+\frac{3}{\pi}\left[\frac{1}{\pi}x^2\sin(\pi x)-\frac{2}{\pi}(-\frac{1}{\pi}x\cos(\pi x)+\frac{1}{\pi^2}\sin(\pi x))\right]+C.
\end{aligned}$$

So,

$$\int_0^1 x^3\sin(\pi x)\,dx = \frac{1}{\pi}-\frac{6}{\pi^3}.$$

Finally,

$$\begin{aligned}
&R[\phi(x)]\\
=&\frac{2\pi^2/9+\pi^2/9+\pi^2/2+\pi^2/45+1/6-1/(4\pi^2)+2\pi/3(1/\pi-6/\pi^3)}{\pi^2/27+1/2+2/3}\\
=&\frac{8.84671318}{1.53220757}=5.77383
\end{aligned}$$

Therefore, the smaller one of the above two Rayleigh quotients should serve as a more accurate upper bound of the smallest eigenvalue

$$\lambda_1 \le 5.72269.$$

3. **Nonuniform vibrating string.** The 1D wave equation on a nonuniform string can be written as

$$u_{tt}=c^2(x)u_{xx}$$

with

$$\text{BC: } u(0,t)=u(L,t)=0,$$

$$\text{IC: } u(x,0)=f(x),\ u_t(x,0)=g(x).$$

(a) By separation of variables, derive the associated Sturm-Liouvill problem.

(b) Show the self-adjointness of the Sturm-Liouville problem you obtain in (a).

(c) Assume the Sturm-Liouville problem has eigenvalues λ_n and associated eigenfunction $\phi_n(x)$. Find a solution formula for $u(x,t)$. Include the initial condition in your formula. (Note: the weight function $\sigma(x) = \frac{1}{c^2(x)}$.)

4. (**Integration by parts**) Physical intuition tells us that if an object has fixed temperature on the boundary and contains no thermal source/sink inside, then by heat diffusion the overall temperature should tend to an equilibrium. Let's us study this for a 1D thin rod with boundary condition of type 1,

$$\frac{\partial}{\partial t}u(x,t) = k\frac{\partial^2}{\partial x^2}u(x,t),$$

$$u(0,t) = u(L,t) = 0,$$

$$u(x,0) = f(x).$$

Consider the function that measures how far away the over temperature is from zero (notice the square here),

$$V(t) = \int_0^L u^2(x,t)\,dx.$$

Prove that, as long as $u(x,t)$ is not a constant, $V(t)$ always decreases with time, that is,

$$\frac{d}{dt}V(t) < 0 \qquad \text{unless} \quad u(x,t) \equiv \textit{constant}.$$

Hint: plug in the definition of $V(t)$, use the PDE to replace u_t with x derivatives and then apply integration by parts.