Math 454 HW 3

Due: Oct 22 at noon

Problem 1, 2, 5: 20 points each — check details.

Problem 3, 4: 10 points each upon completion.

1. **1D wave equation** – series solution – mixed boundary conditions.

   Consider a vibrating string with one end fixed and one end free. It can be modeled with the 1D wave equation with mixed boundary conditions.

$$u_{tt} = c^2 u_{xx},$$

$$u(0,t) = 0, \quad u_x(L,t) = 0,$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(t).$$

   (a) Find a solution formula in terms of an infinite series. (You may mimic the procedures learned from class and/or text).

   (b) Use (a) to find possible forms of $f(x)$ and $g(x)$ such that the solution is a standing wave with 3 nodes. Plot such a solution (you may pick a simple case) at different times in one single graph and mark the nodes with a $*$.

__**Solution.**__ (a) By separation of variables $u(x,t) = \Phi(x)G(t)$, we have two ODEs

$$G(t) = -c^2\lambda G(t) \Rightarrow$$

$$G(t) = a\cos(c\sqrt{\lambda}t) + b\sin(c\sqrt{\lambda}t)$$

and

$$\Phi''(x) = -\lambda\Phi(x), \text{ with BC's } \Phi(0) = \Phi'(L) = 0.$$

This is a Sturm-Liouville eigenvalue problem. The solutions are

$$\lambda_n = \left(\frac{(n+0.5)\pi}{L}\right)^2, \quad \Phi_n(x) = \sin(\sqrt{\lambda_n}x), \quad n = 0, 1, 2, ... \tag{1}$$

Therefore the general solution is

$$u(x,t) = \sum_{n=0}^{\infty} \sin(\sqrt{\lambda_n}x)\left[a_n\cos(c\sqrt{\lambda_n}t) + b_n\sin(c\sqrt{\lambda_n}t)\right].$$

Apply the BC $u(x,0) = f(x)$,

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} \sin(\sqrt{\lambda_n}x) \cdot a_n$$

$$\rightarrow a_n = \frac{\int_0^L f(x)\sin(\sqrt{\lambda_n}x)dx}{\int_0^L \left(\sin(\sqrt{\lambda_n}x)\right)^2 dx} = \frac{2}{L}\int_0^L f(x)\sin(\sqrt{\lambda_n}x)dx.$$

Apply the BC $u_t(x,0) = g(x)$,

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \sin(\sqrt{\lambda_n}x) \cdot c\sqrt{\lambda_n}b_n$$

$$\rightarrow b_n = \frac{1}{c\sqrt{\lambda_n}} \frac{\int_0^L g(x)\sin(\sqrt{\lambda_n}x)dx}{\int_0^L \left(\sin(\sqrt{\lambda_n}x)\right)^2 dx} = \frac{2}{c\sqrt{\lambda_n}L}\int_0^L g(x)\sin(\sqrt{\lambda_n}x)dx.$$

(b) We known any standing wave solution is of the form

$$\sin(\sqrt{\lambda_n}x)\left[a_n\cos(c\sqrt{\lambda_n}t) + b_n\sin(c\sqrt{\lambda_n}t)\right],$$

and we know that for given n, the number of nodes = the number of zeros of $\sin(\sqrt{\lambda_n}x) = \sin(\frac{(n+0.5)\pi x}{L})$ for $x \in (0, L)$. It is easy to check that the sin function vanishes when

$$x = L/(n+0.5), 2L/(n+0.5), ..., nL/(n+0.5) \quad (note\ x \in (0,L)).$$

So, the desired standing wave solution with 3 nodes is

$$u(x,t) = \sin(\sqrt{\lambda_3}x)\left[a_3\cos(c\sqrt{\lambda_3}t) + b_3\sin(c\sqrt{\lambda_3}t)\right]. \tag{2}$$

(If your $\{\lambda_n\}$ starts with $n = 1$ in (1), the 3's should be replaced by 4, the point being that, in this problem, any standing wave with 3 nodes always comes from $\sin(\frac{3.5\pi x}{L})$).

This also means that all other a_n, b_n in the series should be zero.

Now, to obtain the form of $f(x), g(x)$, we simply set $t = 0$ in (2),

$$\begin{aligned} u(x,0) = f(x) &= \sin(\sqrt{\lambda_3}x) \cdot a_3 \\ &= a_3\sin(\frac{3.5\pi x}{L}), \end{aligned}$$

and take $\frac{\partial}{\partial t}$ on (2) and set $t = 0$,

$$\begin{aligned} u_t(x,0) = g(x) =& \sin(\sqrt{\lambda_3}x) \cdot b_3 \cdot c\sqrt{\lambda_3} \\ =& b_3 \cdot c \cdot \frac{3.5\pi}{L} \sin(\frac{3.5\pi x}{L}). \end{aligned}$$

These are the possible forms of $f(x), g(x)$, where a_3, b_3 are some constants and $\lambda_3 = (\frac{3.5\pi}{L})^2$.

2. **Traveling wave.**
Consider a vibrating string (with fixed ends) that is initially at rest,

$$u_{tt} = c^2 u_{xx},$$

$$u(0,t) = u(L,t) = 0,$$

$$u(x,0) = f(x), \quad u_t(x,0) = 0.$$

Show that
$$u(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right]$$
where $F(x)$ is the odd periodic externsion of $f(x)$. Hints:

- For all x, $F(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$.
- $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$.

Solution. We know that the **series** solution of this wave equation is

$$u(x,t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi x}{L}) \left[A_n \cos(c\frac{n\pi}{L}t) + B_n \sin(c\frac{n\pi}{L}t)\right] \tag{3}$$

where, by the initial conditions, $B_n = 0$ and A_n is such that

$$f(x) = \sum_{n=1}^{\infty} \sin(\frac{n\pi x}{L}) \cdot A_n, \text{ for } x \in (0,L).$$

But since the Fourier sine series gives us the **odd extension** of the original function, we have
$$F(x) = \sum_{n=1}^{\infty} \sin(\frac{n\pi x}{L}) \cdot A_n, \text{ for all } x \tag{4}$$

Now, by using the trig identity, rewrite the series solution (3) as

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} \sin(\frac{n\pi x}{L}) \cdot A_n \cos(c\frac{n\pi}{L}t) \\
&= \frac{1}{2}\sum_{n=1}^{\infty} \sin(\frac{n\pi x}{L} + c\frac{n\pi}{L}t) \cdot A_n + \frac{1}{2}\sum_{n=1}^{\infty} \sin(\frac{n\pi x}{L} - c\frac{n\pi}{L}t) \cdot A_n \\
&= \frac{1}{2}\sum_{n=1}^{\infty} \sin(\frac{n\pi}{L}(x+ct)) \cdot A_n + \frac{1}{2}\sum_{n=1}^{\infty} \sin(\frac{n\pi}{L}(x-ct)) \cdot A_n.
\end{aligned}
$$

Comparing this with (4), we conclude that

$$u(x,t) = \frac{1}{2}[F(x+ct) + F(x-ct)].$$

3. **Derivation of Sturm-Liouville problem.**
Derive the associated Sturm-Liouville problem for the following PDEs. Specify the boundary conditions.

(a) Schrödinger equation in 1D,

$$iu_t = -ku_{xx},$$

$$u(0,t) = u(\pi,t), \quad u_x(0,t) = u_x(\pi,t).$$

(b) Heat equation in a 2D disk with circular symmetry (therefore $u = u(r,t)$ and $u_{\theta\theta} = 0$),

$$u_t = \frac{k(r)}{r}(ru_r)_r,$$

$$u(a,t) = 0, \quad u_r(b,t) = 0.$$

(c) Beam equation,

$$u_{tt} + u_{xxxx} + \epsilon u_t + ku = 0,$$

$$u(0,t) = u(1,t) = 0,$$

$$u_{xx}(0,t) = u_{xx}(1,t) = 0.$$

Solution. (a) $\phi'' + \lambda\phi = 0, \ \phi(0) = \phi(\pi), \ \phi'(0) = \phi'(\pi)$.

(b) Let $u(r,t) = R(r)T(t)$. Then

$$RT' = \frac{k(r)}{r}\frac{d}{dr}(rR'T).$$

Divide with RT,

$$\frac{T'}{T} = \frac{k(r)}{rR}(rR')' = -\lambda.$$

Therefore the S-L problem is

$$(rR')' + \lambda \frac{r}{k(r)} R = 0, \; R(a) = R'(b) = 0.$$

(c) Let $u(x,t) = X(x)T(t)$. Then

$$XT'' + X^{(4)}T + \varepsilon XT' + kXT = 0.$$

Divide with kXT and rearrange

$$\frac{T'' + \varepsilon T'}{kT} + 1 = -\frac{X^{(4)}}{X} = -\lambda.$$

Therefore the S-L problem is

$$X^{(4)} - \lambda X = 0, \; X(0) = X(1) = X''(0) = X''(1) = 0.$$

4. **Self-adjoint operator.**
The Sturm-Liouville problem associated with a circular membrane takes the following form

$$r(rf_r)_r + (\lambda r^2 - m^2)f = 0$$

where m is an integer. Prescibe the following boundary conditions

$$f(a) = 0, \quad |f(0)| < \infty, \quad \lim_{r\to 0} rf'(r) = 0.$$

First, rewrite this equation as a standard Sturm-Livioulle problem and then show the self-adjointness of the associated operator.

Solution. Write the equation as a standard S-L problem by dividing it with r,

$$(rf_r)_r - \frac{m^2}{r} f + \lambda r f = 0.$$

Thus the associated linear operator is

$$L[f] = (rf_r)_r - \frac{m^2}{r} f.$$

Now Compute

$$\begin{aligned}\int_0^a L[f]g\,dr &= \int_0^a (rf_r)_r g - \frac{m^2}{r} fg \\ &= -\int_0^a rf_r g_r + rf_r g\Big|_0^a - \int_0^r \frac{m^2}{r} fg \text{ ... integration by parts} \\ &= -\int_0^a rf_r g_r - \int_0^r \frac{m^2}{r} fg \text{ ... boundary conditions}\end{aligned}$$

Now, by switiching f with g in the above equation, we have

$$\int_0^a L[g]f\,dr = -\int_0^a rg_r f_r - \int_0^r \frac{m^2}{r} gf$$

which implies self-adjointness of L, $\int_0^a L[f]g\,dr = \int_0^a L[g]f\,dr$

5. **Orthogonal eigenfunctions.**
Consider a Sturm-Liouville problem

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q(x)\phi + \lambda\sigma(x)\phi = 0$$

with boundary conditions

$$\phi(1) = 0, \quad \phi(2) - 2\phi'(2) = 0.$$

Here p, q, σ are given functions and $p > 0, \sigma > 0$.

(a) Prove the self-adjointness of this problem.

(b) Prove the orthogonality condition for the eigenfunctions. (Hint: notice the weight function σ.)

(c) Derive a formula for a_n in the generalized Fourier series

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x).$$

(Hint: notice the weight function σ.)

Solution. (a) Let's consider a more general form of BC (which includes BC of type 1, type 2 and mixed type whereas periodic BC should be treated a little differently)

$$\alpha\phi(1) + \beta\phi'(1) = 0, \quad \gamma\phi(2) + \delta\phi'(2) = 0. \tag{5}$$

Here, the constants are such that $(\alpha, \beta) \neq (0,0)$, $(\gamma, \delta) \neq (0,0)$.

The operator of interest is

$$L[\phi] = \frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q(x)\phi.$$

Take any two smooth testing functions $u(x), v(x)$ that satisfy BC (5) but are not necessarily the solutions to the S-L problem. We need to verify that $\int_1^2 uL[v] - vL[u]\,dx = 0$. Proceed as follows. First, the terms with $q(x)$ cancel each other for the obvious reason,

$$\int_1^2 uL[v] - vL[u]\,dx = \int_1^2 u\left(\frac{d}{dx}\left[p(x)\frac{dv}{dx}\right]\right) - v\left(\frac{d}{dx}\left[p(x)\frac{du}{dx}\right]\right)\,dx.$$

Then, by the Green's identity, the above integral is reduced to some boundary terms

$$..... = \left(u(x)p(x)\frac{dv(x)}{dx} - v(x)p(x)\frac{du(x)}{dx}\right)\Bigg|_{x=1}^{x=2}$$

That is,

$$\int_1^2 uL[v] - vL[u]\,dx = p(2)\cdot(u(2)v'(2) - v(2)u'(2)) - p(1)\cdot(u(1)v'(1) - v(1)u'(1))\,. \tag{6}$$

Since both $u(x), v(x)$ satisfy BC (5) at $x = 2$, we have

$$\begin{cases} \alpha u(2) + \beta u'(2) = 0 \\ \alpha v(2) + \beta v'(2) = 0 \end{cases}$$

Regarding this as a 2-by-2 linear system with "unknowns" α, β, we see that it admits nontrivial solution $(\alpha, \beta) \neq (0,0)$. By Linear Algebra, the determinant of the coefficient matrix has to be zero

$$\det\begin{pmatrix} u(2) & u'(2) \\ v(2) & v'(2) \end{pmatrix} = 0$$

i.e.

$$u(2)v'(2) - v(2)u'(2) = 0.$$

By the same reason, $u(1)v'(1) - v(1)u'(1) = 0$. Thus, equation (6) equals zero, which proves that L is self-adjoint.

(b) Consider two distinct eigenvalues λ_m, λ_n and associated eigenfunctions $\phi_n(x), \phi_m(x)$ such that

$$L[\phi_n] = -\lambda_n \sigma \phi_n \tag{7}$$

$$L[\phi_m] = -\lambda_m \sigma \phi_m \tag{8}$$

Integrate from 1 to 2 such a term: $\phi_m \times (7) - \phi_n \times (8)$

$$\int_1^2 \phi_m L[\phi_n] - \phi_n L[\phi_m] = (\lambda_m - \lambda_n) \int_1^2 \sigma \phi_m \phi_n \, dx$$

The LHS equals zero by the **self-adjointness** of L. And on the RHS, $\lambda_m - \lambda_n \neq 0$ since λ_n, λ_m are chosen to be distinct. Therefore, we have the orthogonality condition

$$\int_1^2 \sigma \phi_m \phi_n \, dx = 0. \tag{9}$$

(c) Given any **fixed** m, multiply the series

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x),$$

with $\sigma(x)\phi_m(x)$ and integrate from 1 to 2,

$$\int_1^2 \sigma(x)\phi_m(x) f(x) \, dx = \sum_1^{\infty} \int_1^2 a_n \sigma(x) \phi_m(x) \phi_n(x) \, dx.$$

By the **orthogonality** condition, all terms on the RHS vanish except the m-th term,

$$\int_1^2 \sigma(x)\phi_m(x) f(x) \, dx = \int_1^2 a_m \sigma(x) \phi_m(x) \phi_m(x) \, dx.$$

Solve for a_m directly,

$$a_m = \frac{\int_1^2 \sigma(x)\phi_m(x) f(x) \, dx}{\int_1^2 \sigma(x) \phi_m^2(x) \, dx}.$$