

Math 454 HW 3

Due: Oct 22 at noon

Problem 1, 2, 5: 20 points each — check details.

Problem 3, 4: 10 points each upon completion.

1. 1D wave equation – series solution – mixed boundary conditions.

Consider a vibrating string with one end fixed and one end free. It can be modeled with the 1D wave equation with mixed boundary conditions.

$$u_{tt} = c^2 u_{xx},$$

$$u(0, t) = 0, \quad u_x(L, t) = 0,$$

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x).$$

- Find a solution formula in terms of an infinite series. (You may mimic the procedures learned from class and/or text).
- Use (a) to find possible forms of $f(x)$ and $g(x)$ such that the solution is a standing wave with 3 nodes. Plot such a solution (you may pick a simple case) at different times in one single graph and mark the nodes with a *.

Solution. (a) By separation of variables $u(x, t) = \Phi(x)G(t)$, we have two ODEs

$$G(t) = -c^2 \lambda G(t) \Rightarrow$$

$$G(t) = a \cos(c\sqrt{\lambda}t) + b \sin(c\sqrt{\lambda}t)$$

and

$$\Phi''(x) = -\lambda \Phi(x), \quad \text{with BC's } \Phi(0) = \Phi'(L) = 0.$$

This is a Sturm-Liouville eigenvalue problem. The solutions are

$$\lambda_n = \left(\frac{(n + 0.5)\pi}{L} \right)^2, \quad \Phi_n(x) = \sin(\sqrt{\lambda_n}x), \quad n = 0, 1, 2, \dots \quad (1)$$

Therefore the general solution is

$$u(x, t) = \sum_{n=0}^{\infty} \sin(\sqrt{\lambda_n}x) \left[a_n \cos(c\sqrt{\lambda_n}t) + b_n \sin(c\sqrt{\lambda_n}t) \right].$$

Apply the BC $u(x, 0) = f(x)$,

$$u(x, 0) = f(x) = \sum_{n=0}^{\infty} \sin(\sqrt{\lambda_n}x) \cdot a_n$$

$$\rightarrow a_n = \frac{\int_0^L f(x) \sin(\sqrt{\lambda_n}x) dx}{\int_0^L (\sin(\sqrt{\lambda_n}x))^2 dx} = \frac{2}{L} \int_0^L f(x) \sin(\sqrt{\lambda_n}x) dx.$$

Apply the BC $u_t(x, 0) = g(x)$,

$$u_t(x, 0) = g(x) = \sum_{n=1}^{\infty} \sin(\sqrt{\lambda_n}x) \cdot c\sqrt{\lambda_n}b_n$$

$$\rightarrow b_n = \frac{1}{c\sqrt{\lambda_n}} \frac{\int_0^L g(x) \sin(\sqrt{\lambda_n}x) dx}{\int_0^L (\sin(\sqrt{\lambda_n}x))^2 dx} = \frac{2}{c\sqrt{\lambda_n}L} \int_0^L g(x) \sin(\sqrt{\lambda_n}x) dx.$$

(b) We know any standing wave solution is of the form

$$\sin(\sqrt{\lambda_n}x) \left[a_n \cos(c\sqrt{\lambda_n}t) + b_n \sin(c\sqrt{\lambda_n}t) \right],$$

and we know that for given n , the number of nodes = the number of zeros of $\sin(\sqrt{\lambda_n}x) = \sin\left(\frac{(n+0.5)\pi x}{L}\right)$ for $x \in (0, L)$. It is easy to check that the sin function vanishes when

$$x = L/(n+0.5), 2L/(n+0.5), \dots, nL/(n+0.5) \quad (\text{note } x \in (0, L)).$$

So, the desired standing wave solution with 3 nodes is

$$u(x, t) = \sin(\sqrt{\lambda_3}x) \left[a_3 \cos(c\sqrt{\lambda_3}t) + b_3 \sin(c\sqrt{\lambda_3}t) \right]. \quad (2)$$

(If your $\{\lambda_n\}$ starts with $n = 1$ in (1), the 3's should be replaced by 4, the point being that, in this problem, any standing wave with 3 nodes always comes from $\sin\left(\frac{3.5\pi x}{L}\right)$).

This also means that all other a_n, b_n in the series should be zero.

Now, to obtain the form of $f(x), g(x)$, we simply set $t = 0$ in (2),

$$u(x, 0) = f(x) = \sin(\sqrt{\lambda_3}x) \cdot a_3$$

$$= a_3 \sin\left(\frac{3.5\pi x}{L}\right),$$

and take $\frac{\partial}{\partial t}$ on (2) and set $t = 0$,

$$\begin{aligned} u_t(x, 0) = g(x) &= \sin(\sqrt{\lambda_3}x) \cdot b_3 \cdot c\sqrt{\lambda_3} \\ &= b_3 \cdot c \cdot \frac{3.5\pi}{L} \sin\left(\frac{3.5\pi x}{L}\right). \end{aligned}$$

These are the possible forms of $f(x), g(x)$, where a_3, b_3 are some constants and $\lambda_3 = (\frac{3.5\pi}{L})^2$.

2. Traveling wave.

Consider a vibrating string (with fixed ends) that is initially at rest,

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \\ u(0, t) &= u(L, t) = 0, \\ u(x, 0) &= f(x), \quad u_t(x, 0) = 0. \end{aligned}$$

Show that

$$u(x, t) = \frac{1}{2} [F(x - ct) + F(x + ct)]$$

where $F(x)$ is the odd periodic extension of $f(x)$. Hints:

- For all x , $F(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$.
- $\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha + \beta) + \sin(\alpha - \beta)]$.

Solution. We know that the **series** solution of this wave equation is

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[A_n \cos\left(c\frac{n\pi}{L}t\right) + B_n \sin\left(c\frac{n\pi}{L}t\right) \right] \quad (3)$$

where, by the initial conditions, $B_n = 0$ and A_n is such that

$$f(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \cdot A_n, \text{ for } x \in (0, L).$$

But since the Fourier sine series gives us the **odd extension** of the original function, we have

$$F(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \cdot A_n, \text{ for all } x \quad (4)$$

Now, by using the trig identity, rewrite the series solution (3) as

$$\begin{aligned} u(x, t) &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \cdot A_n \cos\left(c\frac{n\pi}{L}t\right) \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L} + c\frac{n\pi}{L}t\right) \cdot A_n + \frac{1}{2} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L} - c\frac{n\pi}{L}t\right) \cdot A_n \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}(x + ct)\right) \cdot A_n + \frac{1}{2} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}(x - ct)\right) \cdot A_n. \end{aligned}$$

Comparing this with (4), we conclude that

$$u(x, t) = \frac{1}{2}[F(x + ct) + F(x - ct)].$$

3. Derivation of Sturm-Liouville problem.

Derive the associated Sturm-Liouville problem for the following PDEs. Specify the boundary conditions.

(a) Schrödinger equation in 1D,

$$iu_t = -ku_{xx},$$

$$u(0, t) = u(\pi, t), \quad u_x(0, t) = u_x(\pi, t).$$

(b) Heat equation in a 2D disk with circular symmetry (therefore $u = u(r, t)$ and $u_{\theta\theta} = 0$),

$$u_t = \frac{k(r)}{r}(ru_r)_r,$$

$$u(a, t) = 0, \quad u_r(b, t) = 0.$$

(c) Beam equation,

$$u_{tt} + u_{xxxx} + \epsilon u_t + ku = 0,$$

$$u(0, t) = u(1, t) = 0,$$

$$u_{xx}(0, t) = u_{xx}(1, t) = 0.$$

Solution. (a) $\phi'' + \lambda\phi = 0$, $\phi(0) = \phi(\pi)$, $\phi'(0) = \phi'(\pi)$.

(b) Let $u(r, t) = R(r)T(t)$. Then

$$RT' = \frac{k(r)}{r} \frac{d}{dr}(rR'T).$$

Divide with RT ,

$$\frac{T'}{T} = \frac{k(r)}{rR} (rR')' = -\lambda.$$

Therefore the S-L problem is

$$(rR')' + \lambda \frac{r}{k(r)} R = 0, \quad R(a) = R(b) = 0.$$

(c) Let $u(x, t) = X(x)T(t)$. Then

$$XT'' + X^{(4)}T + \varepsilon XT' + kXT = 0.$$

Divide with kXT and rearrange

$$\frac{T'' + \varepsilon T'}{kT} + 1 = -\frac{X^{(4)}}{X} = -\lambda.$$

Therefore the S-L problem is

$$X^{(4)} - \lambda X = 0, \quad X(0) = X(1) = X''(0) = X''(1) = 0.$$

4. Self-adjoint operator.

The Sturm-Liouville problem associated with a circular membrane takes the following form

$$r(r f_r)_r + (\lambda r^2 - m^2)f = 0$$

where m is an integer. Prescribe the following boundary conditions

$$f(a) = 0, \quad |f(0)| < \infty, \quad \lim_{r \rightarrow 0} r f'(r) = 0.$$

First, rewrite this equation as a standard Sturm-Liouville problem and then show the self-adjointness of the associated operator.

Solution. Write the equation as a standard S-L problem by dividing it with r ,

$$(r f_r)_r - \frac{m^2}{r} f + \lambda r f = 0.$$

Thus the associated linear operator is

$$L[f] = (r f_r)_r - \frac{m^2}{r} f.$$

Now Compute

$$\begin{aligned} \int_0^a L[f]g \, dr &= \int_0^a (rf_r)_r g - \frac{m^2}{r} fg \\ &= - \int_0^a rf_r g_r + rf_r g \Big|_0^a - \int_0^r \frac{m^2}{r} fg \dots \text{integration by parts} \\ &= - \int_0^a rf_r g_r - \int_0^r \frac{m^2}{r} fg \dots \text{boundary conditions} \end{aligned}$$

Now, by switching f with g in the above equation, we have

$$\int_0^a L[g]f \, dr = - \int_0^a rg_r f_r - \int_0^r \frac{m^2}{r} gf$$

which implies self-adjointness of L , $\int_0^a L[f]g \, dr = \int_0^a L[g]f \, dr$

5. Orthogonal eigenfunctions.

Consider a Sturm-Liouville problem

$$\frac{d}{dx} \left[p(x) \frac{d\phi}{dx} \right] + q(x)\phi + \lambda\sigma(x)\phi = 0$$

with boundary conditions

$$\phi(1) = 0, \quad \phi(2) - 2\phi'(2) = 0.$$

Here p, q, σ are given functions and $p > 0, \sigma > 0$.

- Prove the self-adjointness of this problem.
- Prove the orthogonality condition for the eigenfunctions. (Hint: notice the weight function σ .)
- Derive a formula for a_n in the generalized Fourier series

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x).$$

(Hint: notice the weight function σ .)

Solution. (a) Let's consider a more general form of BC (which includes BC of type 1, type 2 and mixed type whereas periodic BC should be treated a little differently)

$$\alpha\phi(1) + \beta\phi'(1) = 0, \quad \gamma\phi(2) + \delta\phi'(2) = 0. \quad (5)$$

Here, the constants are such that $(\alpha, \beta) \neq (0, 0)$, $(\gamma, \delta) \neq (0, 0)$.

The operator of interest is

$$L[\phi] = \frac{d}{dx} \left[p(x) \frac{d\phi}{dx} \right] + q(x)\phi.$$

Take any two smooth testing functions $u(x), v(x)$ that satisfy BC (5) but are not necessarily the solutions to the S-L problem. We need to verify that $\int_1^2 uL[v] - vL[u] dx = 0$. Proceed as follows. First, the terms with $q(x)$ cancel each other for the obvious reason,

$$\int_1^2 uL[v] - vL[u] dx = \int_1^2 u \left(\frac{d}{dx} \left[p(x) \frac{dv}{dx} \right] \right) - v \left(\frac{d}{dx} \left[p(x) \frac{du}{dx} \right] \right) dx.$$

Then, by the Green's identity, the above integral is reduced to some boundary terms

$$\dots = \left(u(x)p(x) \frac{dv(x)}{dx} - v(x)p(x) \frac{du(x)}{dx} \right) \Big|_{x=1}^{x=2}$$

That is,

$$\int_1^2 uL[v] - vL[u] dx = p(2) \cdot (u(2)v'(2) - v(2)u'(2)) - p(1) \cdot (u(1)v'(1) - v(1)u'(1)). \quad (6)$$

Since both $u(x), v(x)$ satisfy BC (5) at $x = 2$, we have

$$\begin{cases} \alpha u(2) + \beta u'(2) = 0 \\ \alpha v(2) + \beta v'(2) = 0 \end{cases}$$

Regarding this as a 2-by-2 linear system with “unknowns” α, β , we see that it admits nontrivial solution $(\alpha, \beta) \neq (0, 0)$. By Linear Algebra, the determinant of the coefficient matrix has to be zero

$$\det \begin{pmatrix} u(2) & u'(2) \\ v(2) & v'(2) \end{pmatrix} = 0$$

i.e.

$$u(2)v'(2) - v(2)u'(2) = 0.$$

By the same reason, $u(1)v'(1) - v(1)u'(1) = 0$. Thus, equation (6) equals zero, which proves that L is self-adjoint.

(b) Consider two distinct eigenvalues λ_m, λ_n and associated eigenfunctions $\phi_n(x), \phi_m(x)$ such that

$$L[\phi_n] = -\lambda_n \sigma \phi_n \quad (7)$$

$$L[\phi_m] = -\lambda_m \sigma \phi_m \quad (8)$$

Integrate from 1 to 2 such a term: $\phi_m \times (7) - \phi_n \times (8)$

$$\int_1^2 \phi_m L[\phi_n] - \phi_n L[\phi_m] = (\lambda_m - \lambda_n) \int_1^2 \sigma \phi_m \phi_n dx$$

The LHS equals zero by the **self-adjointness** of L . And on the RHS, $\lambda_m - \lambda_n \neq 0$ since λ_n, λ_m are chosen to be distinct. Therefore, we have the orthogonality condition

$$\int_1^2 \sigma \phi_m \phi_n dx = 0. \quad (9)$$

(c) Given any **fixed** m , multiply the series

$$f(x) \sim \sum_{n=1}^{\infty} a_n \phi_n(x),$$

with $\sigma(x)\phi_m(x)$ and integrate from 1 to 2,

$$\int_1^2 \sigma(x)\phi_m(x)f(x) dx = \sum_{n=1}^{\infty} \int_1^2 a_n \sigma(x)\phi_m(x)\phi_n(x) dx.$$

By the **orthogonality** condition, all terms on the RHS vanish except the m -th term,

$$\int_1^2 \sigma(x)\phi_m(x)f(x) dx = \int_1^2 a_m \sigma(x)\phi_m(x)\phi_m(x) dx.$$

Solve for a_m directly,

$$a_m = \frac{\int_1^2 \sigma(x)\phi_m(x)f(x) dx}{\int_1^2 \sigma(x)\phi_m^2(x) dx}.$$