Math 454 HW 2

Due: Oct 6 at noon

1. Use separation of variables to find the solution, in the form of an infinite series, of the homogeneous heat conduction problem with mixed boundary conditions:

$$\begin{aligned} \text{PDE:} \quad & \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \\ \text{BCs:} \quad & u(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0, \\ \text{ICs:} \quad & u(x,0) = f(x) \end{aligned}$$

Proceed as follows:

(a) Assume $u(x,t) = \phi(x)G(t)$ and derive the ODEs satisfied by $\phi(x)$ and $G(t)$. You may use other notations such as $u(x,t) = X(x)T(t)$.

Solution. Plug $u(x,t) = \phi(x)G(t)$ into the PDE

$$\phi(x)G'(t) = k\phi''(x)G(t).$$

Divide it with $k\phi(x)G(t)$ to get

$$\frac{1}{k}\frac{G'(t)}{G(t)} = \frac{\phi''(x)}{\phi(x)}.$$

Since each side depends on different variables, they have to be equal to a constant, say $-\lambda$. So the ODEs are

$$G(t) = -k\lambda G(t), \quad \phi''(x) = -\lambda\phi(x)$$

(b) Solve the ODEs for $\phi(x)$ and $G(t)$, and determine the allowed values for the separation constant λ.

Solution. We first solve the eigenvalue-eigenfunction problem, that is, the ODE of $\phi(x)$. The BC plays a key role here and is derived upon the original BC: $u(0,t) = 0 \implies \phi(0) = 0$ and $\frac{\partial}{\partial x}u(L,t) = 0 \implies \phi'(L) = 0$. Then, we investigate 3 cases in terms of the positivity of λ.

Case 1: $\lambda < 0$. The general solution is

$$\phi(x) = c_1\cosh(\sqrt{-\lambda}x) + c_2\sinh(\sqrt{-\lambda}x).$$

Apply BC: $\phi(0) = 0 \implies c_1 = 0$. Apply BC: $\phi'(L) = 0 \implies c_2\sqrt{-\lambda}\cosh(\sqrt{-\lambda}L) = 0 \implies c_2 = 0$. Therefore, the case $\lambda < 0$ leads to trivial solution!
Case 2: $\lambda = 0$. The general solution is

$$\phi(x) = c_1 + c_2 x.$$

Apply BC: $\phi(0) = 0 \implies c_1 = 0$. Apply BC: $\phi'(L) = 0 \implies c_2 = 0$. Therefore, the case $\lambda = 0$ also leads to trivial solution!
Case 3: $\lambda > 0$. The general solution is

$$\phi(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

Apply BC: $\phi(0) = 0 \implies c_1 = 0$. Apply BC: $\phi'(L) = 0 \implies c_2\sqrt{\lambda}\cos(\sqrt{\lambda}L) = 0$. Thus, either $c_2 = 0$ or $\lambda = 0$ or $\cos(\sqrt{\lambda}L) = 0$. But the first two cases all yield trivial solutions. Therefore, $\sqrt{\lambda}L$ should be equal to a zero of the cosine function i.e. *odd* multiple of $\pi/2$,

$$\sqrt{\lambda}L = (2n-1)\frac{\pi}{2}, \implies \lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2 \quad n = 1, 2, 3, ...$$

Here, we neglect $n \leq 0$ since it would produce the same values due to the square. The associated eigenfunction for each fixed n is

$$\phi_n(x) = \sin\left(\frac{(2n-1)\pi x}{2L}\right).$$

<u>Secondly, we proceed to solve the ODE of $G(t)$, knowing the sequence of eigenvalues λ_n.</u> For each fixed n, the solution to $G'(t) = -k\lambda_n G(t)$ is $G_n(t) = e^{-k\lambda_n t}$ up to a constant multiplier.

(c) Show that the eigenfunctions of the spatial eigenvalue-eigenfunction problem are mutually orthogonal.

<u>Solution.</u> For any indexes $n \neq m$, we want to verify that $\int_0^L \phi_n(x)\phi_m(x)\,dx = 0$. This can be done using the trigonometric identity $\sin\alpha\sin\beta = \frac{1}{2}\cos(\alpha -$

$\beta) - \frac{1}{2}\cos(\alpha+\beta)$

$$
\begin{aligned}
\text{(For } n \neq m\text{)} \quad \int_0^L \phi_n(x)\phi_m(x)\,dx &= \int_0^L \sin\left(\frac{(2n-1)\pi x}{2L}\right) \cdot \sin\left(\frac{(2m-1)\pi x}{2L}\right) dx \\
&= \frac{1}{2}\int_0^L \cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right) dx \\
\text{(Note } n \neq m!\text{)} \qquad &= \frac{L}{2(n-m)\pi}\sin\left(\frac{(n-m)\pi x}{L}\right)\Big|_0^L \\
&\quad - \frac{L}{2(n+m)\pi}\sin\left(\frac{(n+m)\pi x}{L}\right)\Big|_0^L \\
&= 0
\end{aligned}
$$

Then, we want to verify $\int_0^L \phi_n^2(x)\,dx \neq 0$. This can be done by simply using the fact that the integrand $\phi_n^2(x)$ is nonnegative and not uniformly zero.

(d) Write the solution in terms of an infinite series with coefficients B_n, and derive a formula for the B_n in terms of an integral involving the intial condition $u(x,0) = f(x)$.

<u>**Solution.**</u> By the Principle of Superposition, we write the general solution as a linear combination of all the product solutions $\phi_n(x)G_n(t)$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{(2n-1)\pi x}{2L}\right) e^{-k\left(\frac{(2n-1)\pi}{2L}\right)^2 t}$$

Set $t = 0$,

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{(2n-1)\pi x}{2L}\right).$$

Note that this is not a Fourier sine series in the classical sense. Nevertheless, the **orthogonality condition verified in part (c)** validates the coefficient formula

$$B_n = \frac{\int_0^L f(x)\sin\left(\frac{(2n-1)\pi x}{2L}\right) dx}{\int_0^L \sin\left(\frac{(2n-1)\pi x}{2L}\right)^2 dx}$$

Further calculate the denominator,

$$\int_0^L \sin\left(\frac{(2n-1)\pi x}{2L}\right)^2 dx = \int_0^L \frac{1}{2} - \frac{1}{2}\cos\left(\frac{2(2n-1)\pi x}{2L}\right) dx = \frac{L}{2}$$

2. Using separation of variables and the principle of superposition to solve the following boundary value problem:

$$\begin{aligned} u_{xx} + u_{yy} &= 0\,, \qquad \text{for } (x,y) \in (0,L)\times(0,H), \\ u(0,y) &= f(y)\,, \\ u(L,y) &= 0\,, \\ u(x,0) &= g(x)\,, \\ u(x,H) &= 0\,. \end{aligned}$$

Solution. Since this Laplace equation is a linear, homogeneous PDE, the Principle of Superposition applies. To be more specific, we can split the equation into two, each of which handles a nonzero BC (either $f(y)$ or $g(x)$)

$$\text{PDE 1:} \qquad \frac{\partial^2}{\partial x^2}u_1 + \frac{\partial^2}{\partial y^2}u_1 = 0$$

$$\text{with BC} \qquad u_1(0,y) = f(y) \qquad \text{and} \qquad u_1(L,y) = 0$$

$$u_1(x,0) = 0 \qquad \text{and} \qquad u_1(x,H) = 0$$

$$\text{PDE 2:} \qquad \frac{\partial^2}{\partial x^2}u_2 + \frac{\partial^2}{\partial y^2}u_2 = 0$$

$$\text{with BC} \qquad u_2(0,y) = 0 \qquad \text{and} \qquad u_2(L,y) = 0$$

$$u_2(x,0) = g(x) \qquad \text{and} \qquad u_2(x,H) = 0$$

And the final solution will be $u(x,t) = u_1(x,t) + u_2(x,t)$.
To illustrate the solution process, we only look at u_1 (knowing that solving for u_2 follows the same steps except the role of x and y should be switched).
By separation of variables, $u_1(x,y) = X(x)Y(y)$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \lambda$$

Notice that for u_1, the BC at $y=0$ and $y=H$ is homogeneous, suggesting that the y-direction be associated with the eigenvalue problem. Therefore, we solve the Y equation first, which will reveal a family of values for λ as well.

$$-\frac{Y''(y)}{Y(y)} = \lambda, \qquad \text{with BC} \qquad Y(0)=0,\ Y(H)=0.$$

This is standard eigenvalue problem so let me just skip the steps and give the solution to it

$$\lambda_n = \left(\frac{n\pi}{H}\right)^2, \qquad Y_n(y) = \sin\left(\frac{n\pi y}{H}\right), \quad n = 1, 2, 3, ...$$

Now to solve for X, we fix λ_n,

$$\frac{X''(x)}{X(x)} = \lambda_n = \left(\frac{n\pi}{H}\right)^2$$

The characteristic equation is $r^2 - \left(\frac{n\pi}{H}\right)^2 = 0$ and the roots are real $r = \pm\frac{n\pi}{H}$. So

$$X_n(x) = a_n \sinh\frac{n\pi x}{H} + b_n \cosh\frac{n\pi x}{H}$$

Finally the general solution is a linear combination of the product solutions $X_n(x)Y_n(y)$

$$u_1(x, y) = \sum_{n=1}^{\infty}\left(a_n \sinh\frac{n\pi x}{H} + b_n \cosh\frac{nx\pi}{H}\right)\sin\left(\frac{n\pi y}{H}\right)$$

Apply the other BC (as if they were initial condition),

$$u_1(0, y) = f(y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi y}{H}\right)$$

$$\implies \qquad b_n = \frac{2}{L}\int_0^H f(y)\sin\left(\frac{n\pi y}{H}\right)dy$$

$$u_1(L, y) = 0 = \sum_{n=1}^{\infty}\left(a_n \sinh\frac{n\pi L}{H} + b_n \cosh\frac{n\pi L}{H}\right)\sin\left(\frac{n\pi y}{H}\right)$$

$$\implies \qquad a_n \sinh\frac{n\pi L}{H} + b_n \cosh\frac{n\pi L}{H} = 0 \implies \qquad a_n = ...$$

3. Fourier sine and cosine series. **Use graphs to illustrate your answers.**

   (a) Find the Fourier **sine** series for $f(x) = 1 - 3x$ on $x \in (0, 1)$. What value does this series converge to at $x = 0$?

   Solution. Here, we identify that $L = 1$, so the FSS is $f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$

where the coefficient

$$\begin{aligned}
b_n =& \frac{2}{L}\int_0^1 (1-3x)\sin(n\pi x)\,dx \\
\text{(integration by parts)} \qquad =& 2\int_0^1 (1-3x)(-\frac{1}{n\pi})\,d\cos(n\pi x) \\
=& 2\,(1-3x)(-\frac{1}{n\pi})\cos(n\pi x)\Big|_0^1 - 2\int_0^1 (-\frac{1}{n\pi})\cos(n\pi x)\,d(1-3x) \\
=& -\frac{2}{n\pi}\cos(n\pi) + \frac{2}{n\pi} + 0 \\
=& \frac{2}{n\pi}(1-(-1)^n)
\end{aligned}$$

To find the convergence value at $x = 0$, we graph the **odd, periodic extension** of $f(x)$. Then, look near $x = 0$ and check the left limit and right limit of the graph. It turns out the right limit $f(0+) = 1 - 3\cdot 0 = 1$ and the left limit $f(0-) = -1$. So the series converges to $\frac{1}{2}(f(0+) + f(0-)) = 0$ at $x = 0$.

(b) Find the Fourier **cosine** series for $g(x) = 1 - 3x$ on $x \in (0, 1)$. What value does this series converge to at $x = 0$?

<u>**Solution.**</u> Here, we identify that $L = 1$, so the FCS is $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$ where the coefficient

$$a_0 = \frac{1}{L}\int_0^1 g(x)\,dx = x - \frac{3}{2}x^2\Big|_0^1 = -\frac{1}{2} \quad \text{for} \quad n = 0$$

For $n > 0$,

$$\begin{aligned}
a_n =& \frac{2}{L}\int_0^1 (1-3x)\cos(n\pi x)\,dx \\
\text{(integration by parts)} \qquad =& 2\int_0^1 (1-3x)\frac{1}{n\pi}\,d\sin(n\pi x) \\
=& 2\,(1-3x)\frac{1}{n\pi}\sin(n\pi x)\Big|_0^1 - 2\int_0^1 \frac{1}{n\pi}\sin(n\pi x)\,d(1-3x) \\
=& 0 - \frac{6}{(n\pi)^2}\cos(n\pi x)\Big|_0^1 \\
=& \frac{6}{(n\pi)^2}(1-(-1)^n)
\end{aligned}$$

To find the convergence value at $x = 0$, we graph the **even, periodic extension** of $f(x)$. Then, look near $x = 0$ and check the left limit and right limit of the graph. It turns out the graph is continuous at $x = 0$ so the right limit equals the left limit equals the convergence value $g(0) = 1$.

(c) Find the **Fourier series** for

$$h(x) = \begin{cases} 1 - 3x & x \in (0, 1) \\ 2 & x \in (-1, 0] \end{cases}.$$

What value does this series converge to at $x = 0$?

Solution. Here, we identify that $L = 1$, so the FS is

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

where the coefficient

$$a_0 = \frac{1}{2L} \int_{-1}^{1} h(x)\, dx = \frac{1}{2} \int_{-1}^{0} 2\, dx + \frac{1}{2} \int_{0}^{1} 1 - 3x\, dx = \frac{3}{4} \quad \text{for} \quad n = 0$$

For $n > 0$,

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-1}^{1} h(x) \cos(n\pi x)\, dx \\ &= \int_{-1}^{0} 2 \cos(n\pi x)\, dx + \frac{1}{2} \int_{0}^{1} (1 - 3x) \cos(n\pi x)\, dx \\ \text{(integration by parts)} \quad & \ldots\ldots \\ b_n &= \quad \ldots\ldots \end{aligned}$$

To find the convergence value at $x = 0$, we graph the **periodic extension** of $f(x)$ with a period 2. Then, look near $x = 0$ and check the left limit and right limit of the graph. It turns out the right limit $f(0+) = 1 - 3 \cdot 0 = 1$ and the left limit $f(0-) = 2$. So the series converges to $\frac{1}{2}(f(0+) + f(0-)) = 3/2$ at $x = 0$.

4. Using separation of variables, solve Laplace's equation $\nabla^2 u = 0$ inside a $120°$ wedge of radius a, subject to the boundary conditions

$$\begin{aligned} u(r, 0) &= 0\,, \\ u(r, 2\pi/3) &= 0\,, \\ u(a, \theta) &= f(\theta)\,. \end{aligned}$$

Assume a physical condition $|u(0, \theta)| < \infty$.

Solution. One of the key steps in dealing with circular domains (e.g. disks, cylinders) is to use the polar coordinate (r, θ). In this case, we write the Laplacian operator ∇^2 as

$$\nabla^2 u(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}u(r,\theta)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}u(r,\theta).$$

Use separation of variables $u(r, \theta) = R(r)\Theta(\theta)$ in the above formula

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rR'(r)\Theta(\theta)\right) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0$$

that is

$$\frac{1}{r}R'(r)\Theta(\theta) + R''(r)\Theta(\theta) = -\frac{1}{r^2}R(r)\Theta''(\theta)$$

Divide the equation with $R(r)\Theta(\theta)$ and multiply with r^2

$$\frac{rR'(r)}{R(r)} + \frac{r^2R''(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda$$

Notice that in the original problem, the BC at $\theta = 0$ and $\theta = 2\pi/3$ is homogeneous, suggesting that the θ-direction be associated with the eigenvalue problem. Therefore, we solve the Θ equation first, which will reveal a family of values for λ as well.

$$-\frac{\Theta''(\theta)}{\Theta(\theta)} = \lambda, \qquad \text{with BC} \qquad \Theta(0) = 0,\ \Theta(2\pi/3) = 0.$$

This is standard eigenvalue problem so let me just skip the steps and give the solution to it

$$\lambda_n = (3n/2)^2, \qquad \Theta_n(\theta) = \sin(3n\theta/2) \quad n = 1, 2, 3...$$

Now to solve for $R(r)$, we fix λ_n,

$$\frac{rR'(r)}{R(r)} + \frac{r^2R''(r)}{R(r)} = (3n/2)^2$$

This is a so-called *equidimensional* equation where the power of r and the order of derivative is perfectly balanced. The usual technique for solving this type of equation is to "guess" the solution as $R(r) = r^p$ and plug it into the above equation

$$\frac{rpr^{p-1}}{r^p} + \frac{r^2p(p-1)r^{p-2}}{r^p} = (3n/2)^2$$

Cancel out all the powers of r and arrive at

$$p^2 = (3n/2)^2 \qquad \Longrightarrow \; p = \pm 3n/2.$$

Therefore, the solution for $R(r)$ is

$$R_n(r) = B_n r^{3n/2} + C_n r^{-3n/2}.$$

Finally, assemble all the product solutions $\Theta_n(\theta)R_n(r)$

$$\begin{aligned} u(r,\theta) &= \sum_{n=1}^{\infty} \Theta_n(\theta)R_n(r) \\ &= \sum_{n=1}^{\infty} \sin(3n\theta/2)(B_n r^{3n/2} + C_n r^{-3n/2}) \end{aligned}$$

Here, we have *two* families of coefficients to determine. First, use the physical condition at $r = 0$, $|u(0,\theta)| < \infty$,

$$u(0,\theta) = \sum_{n=1}^{\infty} \sin(3n\theta/2) C_n\, 0^{-3n/2} < \infty$$

Since the negative power of 0 is infinity, all the C_n has to vanish to guarantee the above finiteness. Thus,

$$C_n = 0.$$

Finally, use $u(a,\theta) = f(\theta)$ to determine B_n

$$u(a,\theta) = f(\theta) = \sum_{n=1}^{\infty} \sin(3n\theta/2) B_n a^{3n/2}$$

This is a FSS for $\theta \in [0, 2\pi/3]$. So, the n-th coefficient is

$$B_n a^{3n/2} = \frac{2}{2\pi/3} \int_0^{2\pi/3} f(\theta) \sin(3n\theta/2)\, d\theta \qquad \Longrightarrow \qquad B_n = ...$$