Math 454 HW 2

Due: Oct 6 at noon

1. Use separation of variables to find the solution, in the form of an infinite series, of the homogeneous heat conduction problem with mixed boundary conditions:

$$\begin{aligned} \text{PDE:} \quad & \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \\ \text{BCs:} \quad & u(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0, \\ \text{ICs:} \quad & u(x,0) = f(x) \end{aligned}$$

Proceed as follows:

(a) Assume $u(x,t) = \phi(x)G(t)$ and derive the ODEs satisfied by $\phi(x)$ and $G(t)$. You may use other notations such as $u(x,t) = X(x)T(t)$.

Solution. Plug $u(x,t) = \phi(x)G(t)$ into the PDE

$$\phi(x)G'(t) = k\phi''(x)G(t).$$

Divide it with $k\phi(x)G(t)$ to get

$$\frac{1}{k}\frac{G'(t)}{G(t)} = \frac{\phi''(x)}{\phi(x)}.$$

Since each side depends on different variables, they have to be equal to a constant, say $-\lambda$. So the ODEs are

$$G(t) = -k\lambda G(t), \quad \phi''(x) = -\lambda\phi(x)$$

(b) Solve the ODEs for $\phi(x)$ and $G(t)$, and determine the allowed values for the separation constant λ.

Solution. We first solve the eigenvalue-eigenfunction problem, that is, the ODE of $\phi(x)$. The BC plays a key role here and is derived upon the original BC: $u(0,t) = 0 \implies \phi(0) = 0$ and $\frac{\partial}{\partial x}u(L,t) = 0 \implies \phi'(L) = 0$. Then, we investigate 3 cases in terms of the positivity of λ.

Case 1: $\lambda < 0$. The general solution is

$$\phi(x) = c_1\cosh(\sqrt{-\lambda}x) + c_2\sinh(\sqrt{-\lambda}x).$$

Apply BC: $\phi(0) = 0 \implies c_1 = 0$. Apply BC: $\phi'(L) = 0 \implies c_2\sqrt{-\lambda}\cosh(\sqrt{-\lambda}L) = 0 \implies c_2 = 0$. Therefore, the case $\lambda < 0$ leads to trivial solution!

Case 2: $\lambda = 0$. The general solution is

$$\phi(x) = c_1 + c_2 x.$$

Apply BC: $\phi(0) = 0 \implies c_1 = 0$. Apply BC: $\phi'(L) = 0 \implies c_2 = 0$. Therefore, the case $\lambda = 0$ also leads to trivial solution!

Case 3: $\lambda > 0$. The general solution is

$$\phi(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x).$$

Apply BC: $\phi(0) = 0 \implies c_1 = 0$. Apply BC: $\phi'(L) = 0 \implies c_2\sqrt{\lambda}\cos(\sqrt{\lambda}L) = 0$. Thus, either $c_2 = 0$ or $\lambda = 0$ or $\cos(\sqrt{\lambda}L) = 0$. But the first two cases all yield trivial solutions. Therefore, $\sqrt{\lambda}L$ should be equal to a zero of the cosine function i.e. *odd* multiple of $\pi/2$,

$$\sqrt{\lambda}L = (2n-1)\frac{\pi}{2}, \implies \lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2 \quad n = 1, 2, 3, ...$$

Here, we neglect $n \leq 0$ since it would produce the same values due to the square. The associated eigenfunction for each fixed n is

$$\phi_n(x) = \sin\left(\frac{(2n-1)\pi x}{2L}\right).$$

<u>Secondly, we proceed to solve the ODE of $G(t)$, knowing the sequence of eigenvalues λ_n.</u> For each fixed n, the solution to $G'(t) = -k\lambda_n G(t)$ is $G(t) = e^{-k\lambda_n t}$ up to a constant multiplier.

(c) Show that the eigenfunctions of the spatial eigenvalue-eigenfunction problem are mutually orthogonal.

(d) Write the solution in terms of an infinite series with coefficients B_n, and derive a formula for the B_n in terms of an integral involving the intial condition $u(x, 0) = f(x)$.

2. Using separation of variables and the principle of superposition to solve the follow-

ing boundary value problem:

$$\begin{aligned} u_{xx} + u_{yy} &= 0\,, \qquad \text{for } (x,y) \in (0,L) \times (0,H), \\ u(0,y) &= f(y)\,, \\ u(L,y) &= 0\,, \\ u(x,0) &= g(x)\,, \\ u(x,H) &= 0\,. \end{aligned}$$

3. Fourier sine and cosine series. **Use graphs to illustrate your answers.**

   (a) Find the Fourier **sine** series for $f(x) = 1 - 3x$ on $x \in (0,1)$. What value does this series converge to at $x = 0$?

   (b) Find the Fourier **cosine** series for $g(x) = 1 - 3x$ on $x \in (0,1)$. What value does this series converge to at $x = 0$?

   (c) Find the **Fourier series** for

$$h(x) = \begin{cases} 1 - 3x & x \in (0,1) \\ 2 & x \in (-1,0] \end{cases}.$$

   What value does this series converge to at $x = 0$?

4. Using separation of variables, solve Laplace's equation $\nabla^2 u = 0$ inside a 120° wedge of radius a, subject to the boundary conditions

$$\begin{aligned} u(r,0) &= 0\,, \\ u(r,2\pi/3) &= 0\,, \\ u(a,\theta) &= f(\theta)\,. \end{aligned}$$

   Assume a physical condition $|u(0,\theta)| < \infty$.