

Math 454 HW 1

Due: Sep 22 at noon (Full score 65')

1. Given any smooth, single variable function $f(s)$, verify that both $u(x, t) = f(x - 2t)$ and $u(x, t) = f(x + 2t)$ solve the 1D wave equation $u_{tt} = 4u_{xx}$.

Solution. (5' for completion) If $u(x, t) = f(x - 2t)$, by the multivariable Chain Rule

$$\frac{\partial}{\partial t}u(x, t) = f'(x - 2t)\frac{\partial}{\partial t}(x - 2t) = -2f'(x - 2t)$$

$$\frac{\partial^2}{\partial t^2}u(x - 2t) = -2\frac{\partial}{\partial t}f'(x - 2t) = -2f''(x - 2t)\frac{\partial}{\partial t}(x - 2t) = 4f''(x - 2t)$$

and likewise

$$\frac{\partial^2}{\partial x^2}u(x - 2t) = f''(x - 2t).$$

Therefore, $u(x, t)$ solves $u_{tt} = 4u_{xx}$. The same argument applies to $u(x, t) = f(x + 2t)$.

2. Find the unique solution $y = y(x)$ for

$$\frac{d^4y}{dx^4} = y, \quad y(0) = 0, \quad y'(0) = 2, \quad y''(0) = 0, \quad y(\pi) = 0.$$

Solution. (10') The characteristic equation of this ODE is

$$r^4 - 1 = 0$$

for which the solutions are $1, -1, i, -i$. Thus, the ODE has general solution

$$y(x) = c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x.$$

(Note: alternatively, one can use $\sinh x, \cosh x$ and/or e^{ix}, e^{-ix} in the above form.)

Apply the boundary conditions

$$y(0) = 0 = c_1 + c_2 + c_3$$

$$y'(0) = 2 = c_1 - c_2 + c_4$$

$$y''(0) = 0 = c_1 + c_2 - c_3$$

$$y(\pi) = 0 = c_1e^\pi + c_2e^{-\pi} - c_3$$

Solve this algebraic system to obtain $c_1 = c_2 = c_3 = 0, c_4 = 2$ and the specific solution is $y(x) = 2\sin x$.

3. Derive the heat equation on an $L_1 \times L_2$ thin plate. Assume there is no heat source and the lateral faces are insulated. The thickness of the plate is given by function $b(x, y)$ which is much less than L_1, L_2 . Use constant ρ to denote the density, c the specific heat and K_0 the thermal conductivity.

Solution. (10' for completion) Let's examine an infinitesimal area $[x, x + \Delta x] \times [y, y + \Delta y]$. The volume of this part of the plate is $b(x, y)\Delta x\Delta y$ and therefore the thermal energy stored therein is

$$c\rho u(x, y, t)b(x, y)\Delta x\Delta y. \quad (1)$$

The change of thermal energy in this volume amounts to the total contribution from heat fluxes on all 4 sides. Denote the heat flux along x direction $\phi_1(x, y, t)$ and y direction $\phi_2(x, y, t)$. Notice that the cross-sectional areas are approximately $b(x, y)\Delta y, b(x + \Delta x, y)\Delta y$ for ϕ_1 and $b(x, y)\Delta x, b(x, y + \Delta y)\Delta x$ for ϕ_2 . So, thermal energy change per unit time is

$$\begin{aligned} &\phi_1(x, y, t)b(x, y)\Delta y - \phi_1(x + \Delta x, y, t)b(x + \Delta x, y)\Delta y + \\ &\phi_2(x, y, t)b(x, y)\Delta x - \phi_2(x, y + \Delta y, t)b(x, y + \Delta y)\Delta x. \end{aligned}$$

Equate the above expression to the time derivative of (1),

$$\begin{aligned} \frac{\partial}{\partial t}c\rho u(x, y, t)b(x, y)\Delta x\Delta y = &(\phi_1(x, y, t)b(x, y) - \phi_1(x + \Delta x, y, t)b(x + \Delta x, y))\Delta y + \\ &(\phi_2(x, y, t)b(x, y) - \phi_2(x, y + \Delta y, t)b(x, y + \Delta y))\Delta x. \end{aligned}$$

Divide this equation with $\Delta x\Delta y$ and take the limit as $\Delta x \rightarrow 0, \Delta y \rightarrow 0$,

$$\frac{\partial}{\partial t}(c\rho u(x, y, t)b(x, y)) = -\frac{\partial}{\partial x}(\phi_1(x, y, t)b(x, y)) - \frac{\partial}{\partial y}(\phi_2(x, y, t)b(x, y)).$$

Finally, using the Fourier's law to relate the heat flux to the temperature gradient (Note that ϕ_1 is along the x direction and ϕ_2 the y direction)

$$\phi_1 = -K_0\frac{\partial}{\partial x}u, \quad \phi_2 = -K_0\frac{\partial}{\partial y}u.$$

So, the heat equation in 2D is

$$\frac{\partial}{\partial t}(ub) = \frac{K_0}{c\rho} \left[\frac{\partial}{\partial x}(b\frac{\partial}{\partial x}u) + \frac{\partial}{\partial y}(b\frac{\partial}{\partial y}u) \right]$$

4. In the above question, what are the boundary conditions if all edges of the plate are insulated? What if the temperature is given by function $g(x, y, t)$ on the edges?

Solution. (5' for completion) If all edges are insulated, then $\phi_1 = \phi_2 = 0$ on the boundary and in terms of $u(x, y, t)$, the boundary conditions are

$$\frac{\partial}{\partial x}u(0, y, t) = \frac{\partial}{\partial x}u(L_1, y, t) = \frac{\partial}{\partial y}u(x, 0, t) = \frac{\partial}{\partial y}u(x, L_2, t) = 0.$$

If the temperatures are given on the boundary, then the boundary conditions are

$$u(x, y, t) = g(x, y, t) \text{ on the boundary } \{x \in [0, L_1], y = 0\} \cup \{x \in [0, L_1], y = L_2\} \\ \cup \{x = 0, y \in [0, L_2]\} \cup \{x = L_1, y \in [0, L_2]\}$$

5. Consider the modified heat equation

$$u_t = ku_{xx} - \alpha u$$

that models the heat transfer in a 1D rod through which heat is leaking through the lateral sides at rate $\alpha > 0$. Let the boundary conditions be $u(0, t) = 0$ and $u(L, t) = 0$.

- (a) Find the equilibrium temperature distribution.

Solution. (5' for completion) Drop the time derivative and arrive at an ODE

$$ku_{xx} - \alpha u = 0 \text{ with BC } u(0) = u(L) = 0$$

The characteristic equation for this ODE is

$$kr^2 - \alpha = 0$$

Since both k and α are positive constants, the roots are real

$$r_1 = \sqrt{\alpha/k}, r_2 = -\sqrt{\alpha/k}$$

and the general solution is

$$u = c_1e^{r_1x} + c_2e^{r_2x}$$

Apply the BC,

$$u(0) = 0 = c_1 + c_2 \quad \rightarrow \quad c_1 = -c_2 \\ u(L) = 0 = c_1e^{r_1L} + c_2e^{r_2L} = c_1(e^{\sqrt{\alpha/k}L} - e^{-\sqrt{\alpha/k}L})$$

Thus, $c_1 = c_2 = 0$ and the equilibrium solution is $u \equiv 0$.

- (b) Use separation of variables to solve for $u(x, t)$. Assume the initial condition is $u(x, 0) = f(x)$.

Solution. (20')

- Step 1. Look for product solution of the special form

$$u(x, t) = X(x)T(t).$$

Plug this form into the PDE and, upon rearrangement, obtain

$$\frac{1}{k} \frac{T'}{T} + \alpha = \frac{X''}{X}$$

(or, alternatively, $\frac{1}{k} \frac{T'}{T} = \frac{X''}{X} - \alpha$. Values of λ will change correspondingly below.) Since each sides depend on different variables only, they must be both equal to a constant, say, $-\lambda$.

Two ODEs with λ to be determined

$$\frac{1}{k} \frac{T'}{T} + \alpha = -\lambda \quad \rightarrow \quad T(t) = ce^{-k(\alpha+\lambda)t}$$

$$\frac{X''}{X} = -\lambda \text{ together with BC } X(0)=X(L)=0$$

\rightarrow ...skipping discussion on $\lambda > 0, = 0, < 0$...

$$\rightarrow \lambda_n = (n\pi/L)^2, \phi_n(x) = \sin(n\pi x/L), n = 1, 2, 3, \dots$$

So, we found a family of product solutions in the form of

$$e^{-k(\alpha+(n\pi/L)^2)t} \sin(n\pi x/L), n = 1, 2, 3, \dots$$

- By the Principle of Superposition, the general solution is

$$u(x, t) = \sum_{n=1}^{\infty} c_n e^{-k(\alpha+(n\pi/L)^2)t} \sin(n\pi x/L)$$

- The coefficients c_n are determined using the initial condition

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L).$$

This is the Fourier sine series of $f(x)$ on $x \in [0, L]$, so the coef are given by the formula

$$c_n = \frac{\int_0^L f(x) \sin(n\pi x/L) dx}{\int_0^L \sin^2(n\pi x/L) dx} = \frac{2}{L} \int_0^L \sin(n\pi x/L) f(x) dx.$$

- (c) Take the limit as $t \rightarrow \infty$ of your answer to (b). Do you recover your result from (a)?

Solution. (5' for completion) Yes.

$$\begin{aligned} \lim_{t \rightarrow \infty} u(x, t) &= \lim_{t \rightarrow \infty} \sum_{n=1}^{\infty} c_n e^{-k(\alpha + (n\pi/L)^2)t} \sin(n\pi x/L) \\ &= \sum_{n=1}^{\infty} c_n \lim_{t \rightarrow \infty} e^{-k(\alpha + (n\pi/L)^2)t} \sin(n\pi x/L) \\ &= 0 \end{aligned}$$

Note that the exchange of \int_0^T and $\sum_{n=1}^{\infty}$ in the above calculation needs more justification but, for the time being, this is just a formal argument.

6. Is $u_{tt} = c^2(x, y)\nabla^2 u$ a linear PDE of unknown $u(x, y, t)$? What about $u_t + uu_x = 0$? Justify your answers.

Solution. (5' for completion) The 1st PDE is linear since the operator

$$L(u) = \frac{\partial^2}{\partial t^2} u - c^2(x, y) \frac{\partial^2}{\partial x^2} u$$

and it is easy to verify that for any functions u_1, u_2 and constants c_1, c_2

$$\begin{aligned} L(c_1 u_1 + c_2 u_2) &= \frac{\partial^2}{\partial t^2} (c_1 u_1 + c_2 u_2) - c^2(x, y) \frac{\partial^2}{\partial x^2} (c_1 u_1 + c_2 u_2) \\ &= c_1 \frac{\partial^2}{\partial t^2} u_1 + c_2 \frac{\partial^2}{\partial t^2} u_2 - c^2(x, y) c_1 \frac{\partial^2}{\partial x^2} u_1 - c^2(x, y) c_2 \frac{\partial^2}{\partial x^2} u_2 \\ &= c_1 L(u_1) + c_2 L(u_2) \end{aligned}$$

The 2nd PDE is not linear. The operator is

$$L(u) = \frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u$$

Take for example, $u_1(x, t) = t$ and $u_2(x, t) = x$, we already see $L(u_1 + u_2) \neq L(u_1) + L(u_2)$