Math 454 HW 1

Due: Sep 22 at noon (Full score 65')

1. Given any smooth, single variable function $f(s)$, verify that both $u(x,t) = f(x-2t)$ and $u(x,t) = f(x+2t)$ solve the 1D wave equation $u_{tt} = 4u_{xx}$.

   Solution. (5' for completion) If $u(x,t) = f(x-2t)$, by the multivariable Chain Rule

$$\frac{\partial}{\partial t} u(x,t) = f'(x-2t)\frac{\partial}{\partial t}(x-2t) = -2f'(x-2t)$$

$$\frac{\partial^2}{\partial t^2} u(x-2t) = -2\frac{\partial}{\partial t} f'(x-2t) = -2f''(x-2t)\frac{\partial}{\partial t}(x-2t) = 4f''(x-2t)$$

   and likewise

$$\frac{\partial^2}{\partial x^2} u(x-2t) = f''(x-2t).$$

   Therefore, $u(x,t)$ solves $u_{tt} = 4u_{xx}$. The same argument applies to $u(x,t) = f(x+2t)$.

2. Find the unique solution $y = y(x)$ for

$$\frac{d^4 y}{dx^4} = y, \quad y(0) = 0,\ y'(0) = 2,\ y''(0) = 0,\ y(\pi) = 0.$$

   Solution. (10') The characterist equation of this ODE is

$$r^4 - 1 = 0$$

   for which the solutions are $1,\ -1,\ i,\ -i$. Thus, the ODE has general solution

$$y(x) = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x.$$

   (Note: alternatively, one can use $\sinh x,\ \cosh x$ and/or $e^{ix},\ e^{-ix}$ in the above form.) Apply the boundary conditions

$$\begin{aligned} y(0) &= 0 = c_1 + c_2 + c_3 \\ y'(0) &= 2 = c_1 - c_2 + c_4 \\ y''(0) &= 0 = c_1 + c_2 - c_3 \\ y(\pi) &= 0 = c_1 e^{\pi} + c_2 e^{-\pi} - c_3 \end{aligned}$$

   Solve this algebraic system to obtain $c_1 = c_2 = c_3 = 0,\ c_4 = 2$ and the specific solution is $y(x) = 2\sin x$.

3. Derive the heat equation on an $L_1 \times L_2$ thin plate. Assume there is no heat source and the lateral faces are insulated. The thickness of the plate is given by function $b(x, y)$ which is much less than $L_1,\ L_2$. Use constant ρ to denote the density, c the specific heat and K_0 the thermal conductivity.

Solution. (10' for completion) Let's examine an infinitesimal area $[x, x + \Delta x] \times [y, y + \Delta y]$. The volume of this part of the plate is $b(x, y)\Delta x\Delta y$ and therefore the thermal energy stored therein is

$$c\rho u(x, y, t)b(x, y)\Delta x\Delta y. \tag{1}$$

The change of thermal energy in this volume amounts to the total contribution from heat fluxes on all 4 sides. Denote the heat flux along x direction $\phi_1(x, y, t)$ and y direction $\phi_2(x, y, t)$. Notice that the cros-sectional areas are approximately $b(x, y)\Delta y$, $b(x+\Delta x, y)\Delta y$ for ϕ_1 and $b(x, y)\Delta x$, $b(x, y+\Delta y)\Delta x$ for ϕ_2. So, thermal energy change per unit time is

$$\begin{aligned}\phi_1(x, y, t)b(x, y)\Delta y - \phi_1(x + \Delta x, y, t)b(x + \Delta x, y)\Delta y+ \\ \phi_2(x, y, t)b(x, y)\Delta x - \phi_2(x, y + \Delta y, t)b(x, y + \Delta y)\Delta x.\end{aligned}$$

Equate the above expression to the time derivative of (1),

$$\begin{aligned}\frac{\partial}{\partial t}c\rho u(x, y, t)b(x, y)\Delta x\Delta y =&(\phi_1(x, y, t)b(x, y) - \phi_1(x + \Delta x, y, t)b(x + \Delta x, y))\Delta y+ \\ &(\phi_2(x, y, t)b(x, y) - \phi_1(x, y + \Delta y, t)b(x, y + \Delta y))\Delta x.\end{aligned}$$

Divide this equation with $\Delta x\Delta y$ and take the limit as $\Delta x \to 0$, $\Delta y \to 0$,

$$\frac{\partial}{\partial t}(c\rho u(x, y, t)b(x, y)) = -\frac{\partial}{\partial x}(\phi_1(x, y, t)b(x, y)) - \frac{\partial}{\partial y}(\phi_2(x, y, t)b(x, y)).$$

Finally, using the Fourier's law to relate the heat flux to the temperature gradient (Note that ϕ_1 is along the x direction and ϕ_2 the y direction)

$$\phi_1 = -K_0\frac{\partial}{\partial x}u, \quad \phi_2 = -K_0\frac{\partial}{\partial y}u.$$

So, the heat equation in 2D is

$$\frac{\partial}{\partial t}(ub) = \frac{K_0}{c\rho}\left[\frac{\partial}{\partial x}(b\frac{\partial}{\partial x}u) + \frac{\partial}{\partial y}(b\frac{\partial}{\partial y}u)\right]$$

4. In the above question, what are the boundary conditions if all edges of the plate are insulated? What if the temperature is given by function $g(x, y, t)$ on the edges?

Solution. (5' for completion) If all edges are insulated, then $\phi_1 = \phi_2 = 0 \quad \text{on the boundary}$ and in terms of $u(x, y, t)$, the boundary conditions are

$$\frac{\partial}{\partial x}u(0, y, t) = \frac{\partial}{\partial x}u(L_1, y, t) = \frac{\partial}{\partial y}u(x, 0, t) = \frac{\partial}{\partial y}u(x, L_2, t) = 0.$$

If the temperatures are given on the boundary, then the boundary conditions are

$$u(x, y, t) = g(x, y, t) \text{ on the boundary } \{x \in [0, L_1], y = 0\} \cup \{x \in [0, L_1], y = L_2\}$$
$$\cup \{x = 0, y \in [0, L_2]\} \cup \{x = L_1, y \in [0, L_2]\}$$

5. Consider the modifed heat equation

$$u_t = ku_{xx} - \alpha u$$

that models the heat transfer in a 1D rod through which heat is leaking through the lateral sides at rate $\alpha > 0$. Let the boundary conditions be $u(0, t) = 0$ and $u(L, t) = 0$.

(a) Find the equilibrium temperature distribution.

Solution. (5' for completion) Drop the time derivative and arrive at an ODE

$$ku_{xx} - \alpha u = 0 \text{ with BC } u(0) = u(L) = 0$$

The characteristic equation for this ODE is

$$kr^2 - \alpha = 0$$

Since both k and α are positive constants, the roots are real

$$r_1 = \sqrt{\alpha/k},\ r_2 = -\sqrt{\alpha/k}$$

and the general solution is

$$u = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

Apply the BC,

$$u(0) = 0 = c_1 + c_2 \quad \rightarrow \quad c_1 = -c_2$$
$$u(L) = 0 = c_1 e^{r_1 L} + c_2 e^{r_2 L} = c_1(e^{\sqrt{\alpha/k}L} - e^{-\sqrt{\alpha/k}L})$$

Thus, $c_1 = c_2 = 0$ and the equilibrium solution is $u \equiv 0$.

(b) Use separation of variables to solve for $u(x,t)$. Assume the initial condition is $u(x,0) = f(x)$.

Solution. (20')

- Step 1. Look for product solution of the special form

$$u(x,t) = X(x)T(t).$$

Plug this form into the PDE and, upon rearrangement, obtain

$$\frac{1}{k}\frac{T'}{T} + \alpha = \frac{X''}{X}$$

(or, alternatively, $\frac{1}{k}\frac{T'}{T} = \frac{X''}{X} - \alpha$. Values of λ will change correspondingly below.) Since each sides depend on different variables only, they must be both equal to a constant, say, $-\lambda$.

Two ODEs with λ to be determined

$$\frac{1}{k}\frac{T'}{T} + \alpha = -\lambda \quad \rightarrow \quad T(t) = ce^{-k(\alpha+\lambda)t}$$

$$\frac{X''}{X} = -\lambda \text{ together with BC X(0)=X(L)=0}$$

$$\rightarrow \text{...skipping discussion on } \lambda > 0, = 0, < 0\text{...}$$

$$\rightarrow \quad \lambda_n = (n\pi/L)^2,\ \phi_n(x) = \sin(n\pi x/L),\ n = 1, 2, 3, ...$$

So, we found a family of product solutions in the form of

$$e^{-k(\alpha+(n\pi/L)^2)t}\sin(n\pi x/L),\ n = 1, 2, 3...$$

- By the Principle of Superposition, the general solution is

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-k(\alpha+(n\pi/L)^2)t}\sin(n\pi x/L)$$

- The coefficients c_n are determined using the initial condition

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x/L).$$

This is the Fourier sine series of $f(x)$ on $x \in [0, L]$, so the coef are given by the formula

$$c_n = \frac{\int_0^L f(x)\sin(n\pi x/L)\,dx}{\int_0^L \sin^2(n\pi x/L)\,dx} = \frac{2}{L}\int_0^L \sin(n\pi x/L)f(x)\,dx.$$

(c) Take the limit as $t \to \infty$ of your answer to (b). Do you recover your result from (a)?

Solution. (5' for completion) Yes.

$$\begin{aligned}\lim_{t\to\infty} u(x,t) =& \lim_{t\to\infty} \sum_{n=1}^{\infty} c_n e^{-k(\alpha+(n\pi/L)^2)t} \sin(n\pi x/L) \\ =& \sum_{n=1}^{\infty} c_n \lim_{t\to\infty} e^{-k(\alpha+(n\pi/L)^2)t} \sin(n\pi x/L) \\ =& 0\end{aligned}$$

Note that the exchange of $\int_0^T$ and $\sum_{n=1}^{\infty}$ in the above calculation needs more justification but, for the time being, this is just a formal argument.

6. Is $u_{tt} = c^2(x,y)\nabla^2 u$ a linear PDE of unknown $u(x,y,t)$? What about $u_t + uu_x = 0$? Justify your answers.

Solution. (5' for completion) The 1st PDE is linear since the operator

$$L(u) = \frac{\partial^2}{\partial t^2}u - c^2(x,y)\frac{\partial^2}{\partial x^2}u$$

and it is easy to verify that for any functions u_1, u_2 and constants c_1, c_2

$$\begin{aligned}L(c_1u_1 + c_2u_2) =& \frac{\partial^2}{\partial t^2}(c_1u_1 + c_2u_2) - c^2(x,y)\frac{\partial^2}{\partial x^2}(c_1u_1 + c_2u_2) \\ =& c_1\frac{\partial^2}{\partial t^2}u_1 + c_2\frac{\partial^2}{\partial t^2}u_2 - c^2(x,y)c_1\frac{\partial^2}{\partial x^2}u_1 - c^2(x,y)c_2\frac{\partial^2}{\partial x^2}u_2 \\ =& c_1L(u_1) + c_2L(u_2)\end{aligned}$$

The 2nd PDE is not linear. The operator is

$$L(u) = \frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u$$

Take for example, $u_1(x,t) = t$ and $u_2(x,t) = x$, we already see $L(u_1 + u_2) \neq L(u_1) + L(u_2)$