Math 454 – Review

FINAL EXAM: Dec 22nd (as **origianlly** scheduled)!!!

1. Consider the following 1D heat equation with a source term and nonhomogeneous boundary conditions (k is a positive constant),

$$u_t = ku_{xx} + g(x)$$

$$\text{BC: } \frac{\partial}{\partial x}u(0,t) = 0,\ u(L,t) = h(t),$$

$$\text{IC: } u(x,0) = f(x).$$

   Find a solution formula for the original PDE. Show all your work.

2. Show that any eigenvalue λ of the following 2D Sturm-Liouville problem is strictly positive as long as $\alpha\beta > 0$ and $p(x,y) > 0$, $q(x,y) \leq 0$,

$$div(p\nabla\phi) + (q+\lambda)\phi = 0,\ \text{on } \Omega,$$

$$\alpha\phi + \beta\nabla\phi\cdot\vec{n} = 0,\ \text{on } \partial\Omega.$$

3. Consider a quarter of a 3D cylindar

$$\Omega = \{x^2 + y^2 \leq A^2, x \geq 0, y \geq 0,\ z \in [0,H]\}.$$

   and the wave equation

$$u_{tt} = \nabla^2 u \text{ on } \Omega,$$

$$u = 0 \text{ on } \partial\Omega,$$

$$u(x,y,z,0) = 0,$$

$$u_t(x,y,z,0) = f(x,y,z).$$

   (a) State and solve the associated three dimensional eigenvalue problem.

   (b) Find a solution formula for the original equation.

4. Find the Fourier cosine series of

$$f(x) = x - 1,\ \text{for } x \in [0,1].$$

   Then, explain why we can perform term-by-term differentiation on this series.

5. Consider the Poisson's equation on a 2D square with side length 1,

$$\nabla^2 u = f(x, y).$$

The boundary conditions are

$$u(x, 0) = g(x), u(x, 1) = h(x),$$

$$\frac{\partial}{\partial x} u(0, y) = \frac{\partial}{\partial x} u(1, y) = 0.$$

(a) Find the associated Green's function.

(b) Solve the original PDE.

6. Solve the following heat equation for $u(x, t)$ on $x \in (-\infty, \infty)$, $t \in [0, \infty)$ in terms of Fourier Transforms,

$$u_t = k u_{xx} - \alpha u,$$

$$u(x, 0) = f(x).$$

7. Consider a Sturm-Liouville problem

$$(x + 1)\phi''(x) + \lambda\phi(x) = 0,$$

$$\phi(1) = 2\phi'(1), \ \phi'(3) = 0.$$

Let ϕ_n, ϕ_m be two eigenfunctions associated with different eigenvalues. Prove that

$$\int_1^3 \frac{\phi_n(x)\phi_m(x)}{x + 1}\, dx = 0.$$

8. Consider a heat equation with lateral cooling and nonhomogeneous boundary conditions :

$$u_t = k u_{xx} - \alpha u$$
$$u(0, t) = 0$$
$$u(L, t) = T_0$$
$$u(x, 0) = f(x)$$

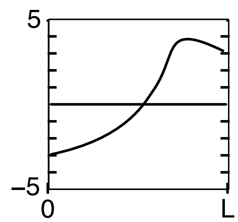
(A) Solve for the equilibrium temperature distribution, $u^{eq}(x)$. Here, α and T_0 are positive constants.

(B) Write a system, including boundary and initial conditions, for $v(x, t) = u(x, t) - u^{eq}(x)$.

(C) Solve for $v(x,t)$. Do **not** simply any integrals. Also, you can write "$u^{eq}(x)$" rather than the solution obtained from (A).

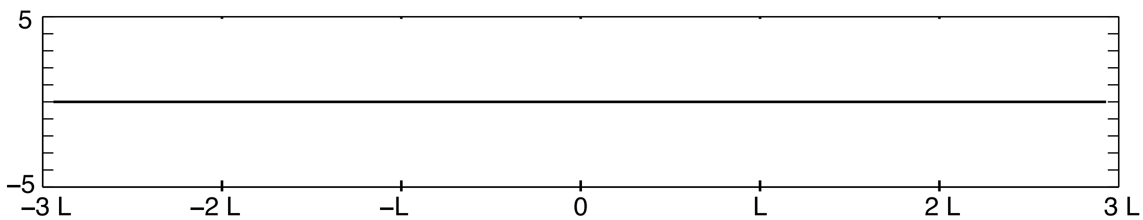
(D) Show $\lim_{t\to\infty} u(x,t) = u^{eq}(x)$.

9. $f(x)$ is defined on $0 \le x \le L$ and has the following graph.



(A) Write the Fourier Cosine Series for $f(x)$, identifying the coefficients.

(B) Sketch the Fourier Cosine Series of $f(x)$ for $-3L \le x \le 3L$. Mark with an * the points at which the FSS doesn't converge to $f(x-)$ or $f(x+)$, if any such points exist.



(C) Is the Fourier Cosine Series of f(x) term-by-term differentiable? Briefly justify your answer.

10. **Traveling wave.**
Consider a vibrating string (with fixed ends) that is initially at rest,

$$u_{tt} = c^2 u_{xx},$$

$$u(0,t) = u(L,t) = 0,$$

$$u(x,0) = f(x), \quad u_t(x,0) = 0.$$

Show that

$$u(x,t) = \frac{1}{2}\left[F(x-ct) + F(x+ct)\right]$$

where $F(x)$ is the odd periodic externsion of $f(x)$.

Hint: $\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$.

11. Derive all the standing wave solutions to the following 1-D vibrating string system with one end fixed and **the other end free**.

$$u_{tt} = c^2 u_{xx}$$
$$u_x(0,t) = 0$$
$$u(\pi,t) = 0$$

(The initial conditions are immaterial to this question).

12. Take the following Sturm-Liouville problem:

$$\phi''(x) + \lambda\phi(x) = 0$$
$$\phi(0) = 0$$
$$\phi(1) = 0$$

   (a) Come up with a reasonable test function for estimating the smallest eigenvalue.
   (b) Estimate the smallest eigenvalue.
   (c) Is your estimate larger, smaller, or equal to the real value?

13. Using the fact that the linear operator for a regular Sturm-Liouville problem is self-adjoint, prove that two eigenfunctions of the operator are orthogonal. You do not have to prove that the operator is self-adjoint, just use that property.

14. Using two functions $u(x)$ and $v(x)$ which obey the same regular Sturm-Liouville boundary conditions, prove that that Sturm-Liouville operator is self-adjoint.

15. Consider a Sturm-Liouville problem

$$\frac{d}{dx}\left[p(x)\frac{d\phi}{dx}\right] + q(x)\phi + \lambda\sigma(x)\phi = 0$$

with boundary conditions

$$\phi'(1) - 2\phi(1) = 0, \quad \phi(2) = 0.$$

Here p, q, σ are given functions and $p(x) > 0, \sigma(x) > 0, q(x) \leq 0$.

   (A) Prove the self-adjointness of this problem. (Hint: apply the Green's formula and handle the boundary terms carefully.)
   (B) Use the Rayleigh's quotient to show that all the eigenvalues are strictly positive.
   (C) State the orthogonality condition for this problem. Please skip the proof.

16. The following PDE system is defined on a half disk such that $0 < r < a$, $0 < \theta < \pi$, $t > 0$.

$$u_t(r,\theta,t) = k\nabla^2 u(r,\theta,t)$$
$$u(a,\theta,t) = 0$$
$$u(r,0,t) = 0$$
$$u(r,\pi,t) = 0$$
$$u(r,\theta,0) = \alpha(r,\theta)$$

(A) State any remaining boundary conditions consistent with the physics.

(B) If a product solution $u(r,\theta,t) = T(t)\phi(r,\theta)$ is posited, then the spatial equation is

$$\nabla^2\phi + \lambda\phi = 0.$$

Solve this eigenvalue problem based on the boundary conditions. Note that the spatial domain is a half-disk and the boundary conditions are NOT periodic.

17. The following PDE system is defined on a **quarter-disk** for $0 \le r \le A$, $0 \le \theta \le \pi/2$.

$$u_t(r,\theta,t) = k\nabla^2 u(r,\theta,t)$$
$$\text{B.C. 1} \qquad u(A,\theta,t) = 0 \text{ for } 0 \le \theta \le \pi/2$$
$$\text{B.C. 2} \qquad u(r,0,t) = 0 \text{ for } 0 \le r \le A$$
$$\text{B.C. 3} \qquad u(r,\pi,t) = 0 \text{ for } 0 \le r \le A$$
$$u(r,\theta,0) = \alpha(r,\theta)$$

(A) Derive the associated 2D Sturm-Liouville problem for $\phi(r,\theta)$. Specify all the boundary conditions. Note that the spatial domain is a quarter-disk and the boundary conditions are **not** periodic.

(B) Show that the eigenvalues and eigenfunctions are

$$\lambda_{mn} = (\frac{z_{mn}}{a})^2,\ \phi_{mn}(r,\theta) = J_m(\frac{z_{mn}}{a}r)\sin(2m\theta)$$

for m = 1, 2, 3,... and n = 1, 2, 3, ... Here, $J_m(z)$ is the 1st kind of Bessel function of order m and z_{mn} $(n = 1,2,3,...)$ are the zeros of $J_m(z)$.

18. Consider a string of length $L = \pi$ with fixed ends. It vibrates under the influence of an external source.

$$u_{tt} = c^2 u_{xx} + e^{-2t}\sin(3x)$$
$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

Assume we know for a fact that the associated Sturm-Liouville problem has a complete set of eigenvalues and eigenfunctions

$$\lambda_n = n^2,\ \phi_n(x) = \sin(nx), \qquad \text{for } n = 0, 1, 2, ...$$

(A) Use eigenfunction expansion on the PDE. Generate a set of second order ODEs for the coefficient functions in the expansion. Simplify all integrals, but do **not** solve the ODEs.

(B) Specify the initial conditions for the ODEs in terms of $f(x)$ and $g(x)$.

19. Prove the Shifting Theorem of Fourier Transformations. That is, if $\mathcal{F}[f(x)] = \hat{f}(\omega)$, then $\mathcal{F}[f(x-c)] = e^{ic\omega}\hat{f}(\omega)$.