Math 454 Handout 3

On Fourier Transform

The Fourier Transform is the analogue of the Fourier series on an infinite domain.

Let's consider first a finite domain $[-L, L]$ with periodic boundary condition. Remember: since $\{e^{\frac{-in\pi x}{L}}\}_{n=-\infty}^{n=\infty}$ solves the Sturm-Liouville problem $\phi'' + \lambda\phi = 0$ with $2L$-periodic boundary conditions, they also form an orthogonal basis for $2L$-periodic functions. Thus, given smooth function $f(x)$, it's Fourier series, in the complex exponential form, is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{-in\pi x}{L}} \tag{1}$$

with coefficients given by

$$c_n = \frac{\int_{-L}^{L} f(x)\overline{e^{\frac{-in\pi x}{L}}}\,dx}{\int_{-L}^{L} |e^{\frac{in\pi}{L}}|^2\,dx} = \frac{1}{2L}\int_{-L}^{L} f(x)e^{\frac{in\pi x}{L}}\,dx \tag{2}$$

Then, we formally let $L \to \infty$ and consider S-L problem $\phi''(x) + \lambda\phi(x) = 0$ for $x \in (-\infty, \infty)$. Then, the solution family consists of $\lambda = \omega^2$ and $\phi_\lambda(x) = \{e^{-i\omega x}\}$ for **all** $\omega \in (-\infty, \infty)$ — thus it's called a continuous spectrum. The Fourier series in (1) become the Fourier integral (just like the limit of Riemann sum becomes an integral)

$$f(x) = \int_{-\infty}^{\infty} c(\omega)e^{-i\omega x}\,d\omega, \tag{3}$$

where the counterpart of (2) is

$$c(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx \tag{4}$$

Definition. Transforms between $f(x)$ and $c(\omega)$ in (3), (4) are called the Fourier transform pair. In particular, $c(\omega)$, more often than not denoted as $\hat{f}(\omega)$, is the Fourier transform of $f(x)$. And, $f(x)$ is the inverse Fourier transform of $\hat{f}(\omega)$.

$$f(x) \leftarrow F.T. \rightarrow \hat{f}(\omega)$$

- Application of Fourier transform in solving PDEs on infinite domains.

Consider the 1D heat equation

$$u_t = ku_{xx} \qquad u(x, 0) = g(x),$$

for $x \in (-\infty, \infty)$. The idea here is to apply FT (4) on each term of the equation and arrive at an ODE of $\hat{f}(\omega)$. Solve for $\hat{f}(x)$ with ease and finally invert it back to $f(x)$ using the inverse FT (3).

So, first, FT the original PDE

$$\hat{u}_t = k\widehat{u_{xx}}$$

(Note that k is a constant and thus can be factored out.) Now, since FT only involves x and ω, it doesn't affect the time derivative on the LSH

$$u_t \leftarrow F.T. \rightarrow \frac{\partial}{\partial t}\hat{u}(\omega, t).$$

But it does affect the RHS.

Differentiation in x-domain amounts to multiplication in ω-domain

In fact, differentiate (3) w.r.t. x

$$\frac{d}{dx}f(x) = \int_{-\infty}^{\infty} c(\omega)\frac{\partial}{\partial x}e^{-i\omega x}\,d\omega = \int_{-\infty}^{\infty} c(\omega)(-i\omega)e^{-i\omega x}\,d\omega$$

and therefore

$$f'(x) \leftarrow F.T. \rightarrow -i\omega\hat{f}(\omega)$$

Do it one more time

$$f''(x) \leftarrow F.T. \rightarrow -\omega^2\hat{f}(\omega).$$

Now, the original PDE, after F.T., becomes

$$\frac{\partial}{\partial t}\hat{u}(\omega, t) = -k\omega^2\hat{u}(\omega, t)$$

which ONLY involves t derivative. In other words, for any fixed ω, the above equation is an ODE and we can simply integrate it to obtain

$$\hat{u}(t, \omega) = \hat{u}(0, \omega)e^{-k\omega^2 t} = \hat{g}(\omega)e^{-k\omega^2 t}.$$

Here, $\hat{u}(0, \omega)$ simply comes from the F.T. of the original initial condition $u(x, 0) = g(x)$.

Finally, apply the inverse F.T. (3) on $\hat{u}$ to get

$$u(x, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{g}(\omega)e^{-k\omega^2 t}e^{-i\omega x}\,d\omega. \tag{5}$$

- Fourier transform and convolution.

The solution formula (5) can be inconvenient sometimes since it involves imaginary numbers and TWO integrations (one for $\hat{g}$ and one for u). To simplify this, we use another very important property of F.T.

Multiplication in ω-domain amounts to convolution in x-domain.

More precisely, if the F.T. of $f(x)$ and $g(x)$ are $\hat{f}(\omega)$ and $\hat{g}(\omega)$, then the inverse F.T. of $\hat{f}(\omega)\hat{g}(\omega)$ is

$$\frac{1}{2\pi} f * g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x_0) g(x - x_0) dx_0 \leftarrow F.T. \rightarrow \hat{f}(\omega)\hat{g}(\omega).$$

(Notice the prefactor $\frac{1}{2\pi}$.) Apply this property to (5), we see that $u(x,t)$ equals the inverse F.T. of $\hat{g}(\omega)e^{-k\omega^2 t}$, which is

$$u(x,t) = \frac{1}{2\pi} g(x) * G(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x_0) G(x - x_0, t)\, dx_0 \tag{6}$$

where $G(x,t)$ is the inverse F.T. of $e^{-k\omega^2 t}$. Classical calculation shows that

$$G(x,t) = \sqrt{\frac{\pi}{kt}} e^{-\frac{x^2}{4kt}}.$$

Final remark: the form of (6) reminds us of the Green's function. In fact, $G(x,t)$ is the Green's function for the heat equation, satisfying

$$G_t = kG_{xx}, \qquad G(x, 0^+) = \delta(x).$$