Math 454 Handout 2

On the Bessel's functions

The Bessel's function is closely related to the Sturm-Liouville problem

$$\nabla^2\phi(r,\theta) + \lambda\phi(r,\theta) = 0 \tag{1}$$

on a 2D circular domain or part of such domain. For simplicity, let's consider a full disk with radius $A\Omega = \{(r,\theta)\,|\,0 \le r \le A,\ -\pi \le \theta \le \pi\}$. The boundary condition is of type 1

$$\phi(A,\theta) = 0.$$

And since $r = 0$ is included in the domain, there should be a (physically implied) finiteness condition at the origin $r = 0$

$$|\phi(0,\theta)| < \infty.$$

Since the range of θ is 2π in this case, there is also a (mathematically implied) periodicity condition along the angular direction

$$\phi(r,-\pi) = \phi(r,\pi), \quad \frac{\partial}{\partial\theta}\phi(r,-\pi) = \frac{\partial}{\partial\theta}\phi(r,\pi).$$

• Remark These implied conditions may change corresponding to the type of domain in consideration.

Solution to the Sturm-Liouville Problem (1)

Step 1. In polar coordinates, the Laplacian ∇^2 has the form

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\phi\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\phi.$$

So the S-L problem is rewritten as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\phi\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\phi = -\lambda\phi.$$

With separation of variables $\phi(r,\theta) = R(r)\Theta(\theta)$, we transform the above equation into

$$\frac{1}{r}\left(rR'\right)'\Theta + \frac{1}{r^2}R\Theta'' = -\lambda R\Theta.$$

Divide it with $\frac{R\Theta}{r^2}$ and rearrange it

$$r\,(rR')'\,\frac{1}{R} + \lambda r^2 = -\frac{\Theta''}{\Theta}.$$

Let both sides equal μ and obtain two ODEs

$$\begin{cases} r\,(rR')'\,\dfrac{1}{R} + \lambda r^2 = \mu \\ \text{with B.C. } R(A) = 0 \text{ and } |R(0)| < \infty \\ \qquad (\text{not necessary if } r = 0 \text{ is not part of the domain. }) \end{cases} \tag{2}$$

$$\Theta'' + \mu\Theta = 0 \text{ with periodic B.C. } \Theta(-\pi) = \Theta(\pi), \Theta'(-\pi) = \Theta'(\pi). \tag{3}$$

Step 2. The Θ equation (3) yields

$$\begin{aligned} &\mu = m^2, \Theta_{m,1}(\theta) = \cos(m\theta), \Theta_{m,2}(\theta) = \sin(m\theta), \text{ for } m = 1, 2, 3, ... \\ &\text{and } \mu = 0, \Theta_0(\theta) = 1. \end{aligned} \tag{4}$$

Note that, due to periodic BC on $\Theta(\theta)$, each positive μ corresponds to two linearly independent eigenfunctions. The result changes if BC is of other types.

Step 3. Fixing $\mu = m^2$ in the R equation (2), we arrive at $r\,(rR')'\,\frac{1}{R} + \lambda r^2 = m^2$, or, equivalently,

$$\frac{d}{dr}\left(r\frac{d}{dr}R(r)\right) + (\lambda r - \frac{m^2}{r})R(r) = 0. \tag{5}$$

This is an S-L problem with self-adjoint operator

$$\mathcal{L}[R] = \frac{d}{dr}\left(r\frac{d}{dr}R(r)\right) - \frac{m^2}{r}R(r),$$

and weight function $\sigma(r) = r$. Moreover, using the Rayleigh quotient, we see $\lambda > 0$. Therefore, perform change of variable $z = \sqrt{\lambda}r$ and $f(z) = R(r)$, we have by the Chain Rule of single-variable Calculus,

$$\frac{d}{dr} = \frac{d}{dz}\frac{dz}{d\lambda} = \frac{d}{dz}\sqrt{\lambda}$$

So the $R(r)$ equation (5) is transformed into the **Bessel's equations of order m** in terms of $f(z)$,

$$\frac{d}{dz}\left(z\frac{d}{dz}f(z)\right) + (z - \frac{m^2}{z})f(z) = 0. \tag{6}$$

In literature, one may see variations of (6), such as (with multiplication of z and product rule)

$$z^2 f''(z) + zf'(z) + (z^2 - m^2)f(z) = 0.$$

Step 4. We know there are two linearly independent solutions, called the **Bessel's function of the 1st kind and 2nd kind** respectively,

$$\begin{cases} \text{1st kind: } J_m(z), \text{ with } |J_m(0)| < \infty, \\ \text{2nd kind: } Y_m(z), \text{ with } |Y_m(0)| = \infty. \end{cases}$$

Therefore, the general solution to (6), the Bessel's equation of order m, is

$$f(z) = c_1 J_m(z) + c_2 Y_m(z).$$

In this particular problem, due to the physical constraint $|R(0)| < \infty$ stated as a B.C. in (2), we only use $J_m(z)$, that is $f(z) = J_m(z)$. Furthermore, since $z = \sqrt{\lambda}\, r$ and $f(z) = R(r)$, the solution to the orignal $R(r)$ equation (5) is

$$R(r) = J_m\left(\sqrt{\lambda}\, r\right).$$

The other B.C. in (2) claims $R(A) = 0$. Therefore, set $r = A$ in the above equation,

$$J_m\left(\sqrt{\lambda} A\right) = 0$$

which implies $\sqrt{\lambda} A$ should a zero of $J_m(z)$. By labeling the zeros of $J_m(z)$ as $z_{m,1}, z_{m,2}, z_{m,3}, \ldots$, we have

$$\sqrt{\lambda_{m,n}} A = z_{m,n} \implies \lambda_{m,n} = \left(\frac{z_{m,n}}{A}\right)^2 \text{ and } R_{m,n}(r) = J_m(\sqrt{\lambda_{m,n}}\, r). \qquad (7)$$

Step 5. All in all, upon combining (4) in Step 2 and (7) in Step 4, the eigenvalues λ and associate eigenfunctions $\phi(r, \theta) = R(r)\Theta(\theta)$ to the original S-L problem is

$$\lambda_{m,n} = \left(\frac{z_{m,n}}{A}\right)^2, \quad \phi_{m,n,1} = \cos(m\theta) J_m(\sqrt{\lambda_{m,n}}\, r), \ \phi_{m,n,2} = \sin(m\theta) J_m(\sqrt{\lambda_{m,n}}\, r),$$

$$\text{for } m = 1, 2, 3..., \ n = 1, 2, 3, ...$$

$$\lambda_{0,n} = \left(\frac{z_{0,n}}{A}\right)^2, \quad \phi_{0,n,1} = J_m(\sqrt{\lambda_{0,n}}\, r), \text{ for } n = 1, 2, 3, ...$$

In the index pair (m, n), m is associated with the solutions (4) to the Θ equation and corresponds to the order of Bessel's equation (6) and its solutions $J_m(z)$, $Y_m(z)$. n comes from the ordering of zeros of $J_m(z)$.

• Remark. If it helps understanding, you may think that the Bessel's functions J_m and Y_m play a similar role as the sine and cosine functions — remember for the S-L problem $\nabla^2\phi(x,y) + \lambda\phi(x,y) = 0$ on an $L \times H$ rectangle with BC of type 1, the eigenfunctions are

$$\phi_{m,n} = \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right).$$

Generalized Fourier Series using Bessel's functions

Given any smooth function $g(r,\theta)$ defined on a full disk with radius A. Let it satisfy the BC

$$g(A,\theta) = 0, \qquad |g(0,\theta)| < \infty.$$

Then, using the eigenfuncitons obtained in Step 5, we can expand $g(r,\theta)$ as a double series

$$g(r,\theta) = \sum_{n=1}^{\infty} c_{0,n}\phi_{m,n,0}(r,\theta) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(c_{m,n}\phi_{m,n,1}(r,\theta) + d_{m,n}\phi_{m,n,2}(r,\theta)\right)$$

where the coefficients are given as

$$c_{m,n} = \frac{\int_{\theta=-\pi}^{\theta=\pi}\int_{r=0}^{r=A} g(r,\theta)\phi_{m,n,1}(r,\theta)r\,drd\theta}{\int_{\theta=-\pi}^{\theta=\pi}\int_{r=0}^{r=A} \phi_{m,n,1}(r,\theta)^2 r\,drd\theta}$$

$$d_{m,n} = \frac{\int_{\theta=-\pi}^{\theta=\pi}\int_{r=0}^{r=A} g(r,\theta)\phi_{m,n,2}(r,\theta)r\,drd\theta}{\int_{\theta=-\pi}^{\theta=\pi}\int_{r=0}^{r=A} \phi_{m,n,2}(r,\theta)^2 r\,drd\theta}$$

• Remark. The weight function r here comes from the S-L problem (5).