Math 354 Homework 8

Due on Friday, April 19th

Notation. We define the characteristic function of interval (a, b) as

$$\chi_{(a,b)}(x) = \begin{cases} 1, & x \in (a, b) \\ 0, & otherwise \end{cases}$$

Problem 1 (wavelets). Define a mother wavelet $\psi = \chi_{(0,0.5)} - \chi_{(0.5,1)}$ and the Haar wavelets $h_{jn} = 2^{j/2}\psi(2^j x - n)$ as a result of dilations and translations of ψ. Also define $h_{(0)}(x) = 1$.

Find the best $L^2([0, 1])$ approximation of $f(x) = 10\chi_{(0,0.2)}$ with the form $c_0 h_{(0)} + c_1 h_{00} + c_2 h_{10} + c_3 h_{11}$. Examine the magnitudes of the coefficients in terms of the localization of the discontinuity of $f(x)$ at $x = 0.2$.

Problem 2 (function space and convolution).

a) Define two functions, $f(x) = x^{-1}\chi_{(1,\infty)}$ and $g(x) = x^{-0.5}\chi_{(0,1)}$. Show that $f \in L^2$ but not L^1 and $g \in L^1$ but not L^2. Here the domain is taken as $(-\infty, \infty)$.

b) Formally, show that $f * (g * h) = (f * g) * h$.

c) If a kernel function $K(x, t)$ satisfies the wave equation $u_{tt} = c^2 u_{xx}$, show (formally) that $u(x, t) = K(x, t) * f(x)$ is also a solution. Here the convolution is taken with respect to the spatial variable x.

Problem 3 (the Fourier Transform).

a) (Table 2, #10) For $a > 0$, show that $\mathcal{F}^{-1}[e^{-a|\xi|}] = \frac{a}{\pi(x^2+a^2)}$.

b) (§7.3, #12a) For $a > 0$, define $f_a(x) = \frac{a}{\pi(x^2+a^2)}$. With the help of Fourier Transform, show that $f_a * f_b = f_{a+b}$ (here $a > 0, b > 0$).

Problem 4 (Partial Differential Equations). Consider an infinite strip $x \in (-\infty, \infty)$, $y \in [0, b]$. Its temperature at the boundaries is held at $f(x)$ and $g(x)$. Then the temperature distribution at equilibrium satisfies the Dirichlet problem

$$u_{xx} + u_{yy} = 0,\ u(x, 0) = f(x),\ u(x, b) = g(x).$$

Solve for $u(x, y)$ in the form of a Fourier integral. Leave the integral as is.

Problem 5 (multivariable Fourier Transform). We know that multiplication in the physical domain amounts to convolution in the frequency domain. But the following is also true. Why?

$$\text{If } f(\vec{x}) = f_1(x_1)f_2(x_2)...f_n(x_n),$$

$$\text{then } \hat{f}(\vec{\xi}) = \hat{f}_1(\xi_1)\hat{f}_2(\xi_2)...\hat{f}_n(\xi_n).$$

Problem 6 (the discrete Fourier Transform). In this problem, we consider the discrete Fourier Transform on $\mathbf{C}^N$ defined as

$$\vec{a} = M\vec{\hat{a}}, \quad \vec{\hat{a}} = M^*\vec{a}$$

where the N-by-N Fourier matrix $M = (M_{jk})$ with $M_{jk} = \frac{1}{\sqrt{N}}(e^{\frac{2\pi i}{N}})^{jk}$ and conjugate transpose $M^* = \overline{M}^T$.

a) Knowing that M is an orthonormal matrix, namely, $MM^* = M^*M = I$, show that $\|\vec{a}\| = \|\vec{\hat{a}}\|$. Here the norm of a vector $\vec{a}$ is defined as $\|\vec{a}\| = \sqrt{\vec{a}^*\vec{a}} = \sqrt{|a_1|^2 + ... + |a_N|^2}$.

b) If $\vec{a} \in \mathbf{R}^N$, show that $\overline{\hat{a}_m} = \hat{a}_{N-m}$.