Math 354 Homework 7

Due at 10am Wednesday, April 2nd

Problem 1. Consider a set of orthogonal polynomials $\{p_n(x)\}_{n=0}^{\infty}$ where $\deg\{p_n\} = n$. Show that $\langle p_n, x^n \rangle \neq 0$.

Problem 2. Here let's examine the behavior of the Legendre polynomials near $x = \pm 1$. Define

$$p_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

a) Use the Leibniz rule – see e.g. Wiki entry: *Leibniz rule (generalized product rule)* – to show that

$$p_n(1) = 1, \qquad p_n(-1) = (-1)^n.$$

Hint: factorize $(x^2 - 1)^n$ into two n-th degree polynomials.

b) Use the differential equation that p_n solves to show that

$$p_n'(1) = \frac{n(n+1)}{2}, \qquad p_n'(-1) = -\frac{n(n+1)}{2}(-1)^n.$$

c) Sketch $p_5(x)$ in the neighborhoods of $x = \pm 1$. Then on a separate graph, sketch $p_{10}(x)$ near $x = \pm 1$.

Problem 3. Let $f(x)$ be the Heaviside function, that is,

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

a) Find the best $L^2([-1, 1])$ approximation of $f(x)$ among all polynomials with degree no more than 3. Denote your answer as $f_3(x)$.

Hint: use the Legendre polynomials.

b) Plot $f_3(x)$ on interval $[-1.5, 1.5]$ using Matlab. **Attach the printout to your solutions.**

Hint: the following command plots $3(x^2 + 1)$ for $0 \leq x \leq 2$,
$>>$ $ezplot('3 * (x\hat{\ }2 + 1)', [0, 2]);$

c) By looking at the computer output, roughly indicate over which interval $f_3(x)$ is of the most faithful approximation to the Heaviside function $f(x)$. Why so?