Math 354 Homework 6
Part II

Due at 10am Friday, March 21st

Problem 1 (heat equation with source term). Here, let's consider a π-long rod with insulated ends. Its thermal diffusivity is $k = 10$. The left half of the rod is being heated with an external source, $2e^{-t}$, while the right half is unaffected. Initially, the temperature is 0.

a) Build a PDE to model the temperature distribution. Specify the source term in your equation.

b) Identify the orthonormal basis that suits the PDE from part a).

c) Represent the PDE using (generalized) Fourier series with **time-dependent** coefficients. Then, with the help of variation of parameters, solve the PDE.

d) What is the asymptotic temperature distribution as $t \to \infty$? Give a brief physical interpretation.

Problem 2 (Bessel's equation). The Bessel's equation is closely related to the polar coordinate system. Following the steps below to derive the Bessel's equation from the 2-D **wave** equation.

a) Consider the Laplace operator in 2-D

$$\Delta = \partial_{xx} + \partial_{yy}.$$

Under change of variables

$$x = r\cos\theta,\ y = r\sin\theta,$$

show that

$$\Delta = \partial_{rr} + r^{-1}\partial_r + r^{-2}\partial_{\theta\theta}.$$

Hint: start with the polar form and apply the chain rule, e.g.

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}.$$

b) In terms of polar coordinates, the 2-D wave equation $u_{tt} = c^2\Delta u$ is of the form

$$u_{tt} = c^2(u_{rr} + r^{-1}u_r + r^{-2}u_{\theta\theta}).$$

Now, use separation of variables by assuming $u(r,\theta,t) = R(r)\Theta(\theta)T(t)$ to derive the following ODEs

$$T'' + c^2\alpha^2 T = 0,$$

$$\Theta'' + \lambda^2\Theta = 0,$$

$$r^2R'' + rR' + (\alpha^2 r^2 - \lambda^2)R = 0,$$

where α, λ are constants.

c) Finally, by change of variable $x = \alpha r$ and correspondingly $f(x) = R(r)$, show that the last equation in part b) amounts to the Bessel's equation

$$x^2 f'' + x f' + (x^2 - \lambda^2) f = 0.$$