Math 354 Homework 6
Part I

Due at 10am Friday, March 21st

Problem 1 (self-adjoint operator).

a) Consider the vector space $\mathcal{R}^n$ of $n \times 1$ real vectors. Let A be a real symmetric $n \times n$ matrix. Define an operator $\mathcal{A}$ as

$$\mathcal{A} : \vec{x} \mapsto A\vec{x} \qquad \text{for any } \vec{x} \in \mathcal{R}^n.$$

Show that $\mathcal{A}$ is a self-adjoint operator with respect to the inner product

$$\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y} = \vec{x}^T \vec{y}.$$

b) Let operator $\mathcal{L}[f] = f''$. Consider two smooth functions $f_1(x), f_2(x)$ that satisfy the same boundary conditions

$$f_i(0) = 0,\ f_i'(\pi) = 0 \qquad \text{for } i = 1, 2.$$

Show that $\mathcal{L}$ is a self-adjoint operator on functions of $L^2([0, \pi])$ that satisfy the above boundary conditions. Namely, show that

$$\langle f_1, \mathcal{L}[f_2] \rangle = \langle \mathcal{L}[f_1], f_2 \rangle.$$

Problem 2. In this problem, we consider a differential operator $\mathcal{L}$ defined as

$$\mathcal{L}[f(x)] = (xf'(x))'.$$

a) Show that the Sturm-Liouville problem

$$\mathcal{L}[f] + \lambda x^{-1} f = 0, \qquad f(0) = f(e) = 0$$

has eigevenvalues and corresponding eigenfunctions

$$\{\lambda_n = (n\pi)^2,\ f_n(x) = \sin(n\pi \log(x))\}_{n=1}^{\infty}.$$

b) Normalize f_n in part a) with respect to the weighted inner product $\langle \cdot, \cdot \rangle_{x^{-1}}$ and associated norm.

c) Expand $f(x) = 1$ in the weighted $L^2_w([0, e])$ space using your results from part b).

Problem 3. Consider a vibrating string in an elastic medium that is modeled as

$$u_{tt} = c^2 u_{xx} - a^2 u.$$

The boundary condition is of mixed type

$$u(0, t) = 0, \ u_x(b, t) = 0.$$

At time $t = 0$, the string is at its neutral position, i.e., $u(x, 0) = 0$ with velocity $u_t(x, 0) = x$. Solve for $u(x, t)$ using all the techniques you've learned so far.