Math 354 Homework 5

Due at 10am Wednesday, March 5th

Problem 1. (§ 3.3 #9) Suppose $\{\phi_n\}$ is an orthonormal basis of $L^2([a,b])$.

a) Show that for any constants $\{b_m\}$,

$$\left\langle \phi_n, \sum_m b_m \phi_m \right\rangle = \overline{b_n}.$$

b) Show that for any constants $\{a_n\}$, $\{b_m\}$,

$$\left\langle \sum_n a_n \phi_n, \sum_m b_m \phi_m \right\rangle = \sum_n a_n \overline{b_n}.$$

c) For any $f, g \in L^2([a,b])$, show that

$$\langle f, g \rangle = \sum_n \langle f, \phi_n \rangle \, \overline{\langle g, \phi_n \rangle}.$$

Problem 2. (§ 3.4 # 1) Show that

$$\left\{ e^{2\pi i(mx+ny)} \right\}_{m,n=-\infty}^{\infty}$$

is an orthonormal set in $L^2([0,1] \times [0,1])$. Here the domain is the unit square enclosed by $x = 0, x = 1, y = 0, y = 1$.

Problem 3. Given an orthonormal basis of $L^2([0,\pi])$,

$$\left\{ \sqrt{\frac{2}{\pi}} \sin(n - \frac{1}{2})x \right\}_{n=1}^{\infty}.$$

a) Find the (generalized) Fourier series of $f(x) = 1$ using this basis.

b) Based on part a), what is the best $L^2([0,\pi])$ approximation of $f(x) = 1$ among all functions of the form

$$a \sin \frac{1}{2}x + b \sin \frac{3}{2}x + c \sin \frac{5}{2}x$$

Problem 4. Let $w(x) = e^{-x}$. Define a weighted L^2 inner product as

$$\langle f, g \rangle_w = \int_0^\infty f(x)\overline{g(x)}e^{-x}\,dx.$$

a) Show that $e^{\alpha x} \in L^2_w([0,\infty])$ as long as constant $\alpha < \frac{1}{2}$.

b) Show that $\{1, x-1\}$ form an orthonormal set in $L^2_w([0,\infty])$. You may use the fact that

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

c) Find the best L^2_w approximation of x^2 among all 1st order polynomials.