Math 354 Homework 3

Due at 10am Friday, Feb. 8th

The following Fourier sine/cosine series may be useful.

$$\theta = 2\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\theta \quad \text{for } \theta \in (0, \pi)$$

$$\theta = \frac{\pi}{2} - \frac{4}{\pi}\sum_{0}^{\infty} \frac{\cos(2n-1)\theta}{(2n-1)^2} \quad \text{for } \theta \in (0, \pi).$$

Problem 1. Consider the heat flow in a π-long rod with insulated end points. Let a positive constant k denote the thermal diffusivity of the rod.

Initially the temperature distribution is

$$u(x, 0) = x, \qquad \text{for } x \in (0, \pi).$$

a) Write a heat equation with proper boundary and initial conditions.

b) Solve this equation (with the help of the series given above). And use your solution to discuss the asymptotic profile of temperature $u(x, t)$ as $t \to +\infty$.

Problem 2. Consider the heat flow in a π-long rod with the temperature being held as 2 at the left endpoint and -3 at the right endpoint. Initially the temperature is uniformly zero on the rod. Study this model as in the previous problem, i.e., build the PDE, find the solution and discuss the asymptotic behavior of the solution.

Problem 3. Consider a square plate with side length π. The sides $x = 0$ and $x = \pi$ are insulated whereas the side $y = 0$ is held at temperature zero and the side $y = \pi$ is held at temperature $u(x, \pi) = 3\cos 2x - 5\cos 3x$. Find the **steady-state** temperature on this plate with the following steps.

a) Build a Laplace equation with proper boundary conditions.

b) Use separation of variables to find a pair of ODEs in terms of the x-variable and the y-variable respectively.

c) Solve the ODEs obtained in part b) and use the boundary condition at $x = 0$ and $x = \pi$ to identify a family of "special" solutions.

d) Take linear combination of the special solutions found in part c), which yields the general solution. Then apply the boundary condition at $y = 0$ and $y = \pi$ to finally solve for $u(x, y)$ which should look like

$$u(x, y) = 3\frac{\sinh 2y}{\sinh 2\pi}\cos 2x - 5\frac{\sinh 3y}{\sinh 3\pi}\cos 3x$$

Show your work!