Math 316 Midterm Exam

Name ______________

Oct 22nd, 2007 6-8pm

No Calculators.

Problems on **both sides**.

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Write by hand and sign the following **honor pledge**:

I pledge on my honor that I have not given or received any unauthorized assistance on this examination.

Signature:______________

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Total score:______________

Problem	1(25')	2(25')	3(25')	4(25')	5(25')	6(25')	7(25')	8(25')	9(extra 10')
Scores									

1. Consider the following equations.

a) (15') $y' = y^2 \cos t, \quad y(0) = 1$. Find the particular solution.

b) (5') $(t^2 + 1)\frac{dy}{dt} + 2yt + 1 = 0$. Is this equation exact and why?

c) (5') $y'' + y \ln|t+1| = \ln|t-2|, \quad y(1) = 2$. Find the largest interval over which the solution exists.

2. Consider an equation

$$y'' + 2y' + 5y = 0, \qquad y(0) = y'(0) = 2.$$

a) (20') Solve for $y(t)$. Specify all constants.

b) (5') Transform the equation into a first order system with initial conditions.

3. Consider a 2×2 system $x' = Ax$ where A is a constant matrix. Let $M(t)$ be a fundamental matrix.

a) (10') Is it possible that $M(t) = \begin{pmatrix} 2e^{2t} & -e^{3t} \\ 4e^{2t} & -2e^{3t} \end{pmatrix}$? Justify your answers.

b) (10') Prove that $M(t)C$ is also a fundamental matrix. Here C is a 2×2 invertible matrix.

c) (5') Write down the definition for e^{At}.

4. The population $y(t)$ of certain species is governed by

$$y' = -Cy(y-K)(y-T)$$

where C, K, T are positive constants. Assume $K > T$.

a) (10') Plot the phase line and direction field.

b) (10') Find all equilibrium solutions and indicate their types.

c) (5') If the initial population is somewhere between T and K, what is the asymptotic behavior of $y(t)$ as $t \to +\infty$?

5. Consider a system $\vec{x}' = A\vec{x}$ where $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$.

a) (15') Find the general solution.

b) (10') Plot a phase portrait and indicate the type of the origin as an equilibrium solution.

6. (25') Consider an equation $y' + y = \sin t + \cos t$. Show that for any solution $y(t)$,

$$\lim_{t \to +\infty} [y(t) - \sin t] = 0.$$

Hint: use the fact that $\hat{y}(t) = \sin t$ is a particular solution.

7. Let $y(t)$ and $z(t)$ be two solutions to $x'' - 2x' + e^{-t}x = 0$ with initial conditions $y(0) = y'(0) = z(0) = 1, z'(0) = 2$.

a) (15') Compute the Wronskian $W[y, z]$ at $t = 2$.

b) (10') Is it possible that y and z share a common zero? Justify your answer.

8. Consider a system $\vec{x}' = A\vec{x} + \vec{g}$.

a) (15') Use fundamental matrix $M(t)$ for system $\vec{y}' = A\vec{y}$ to derive a solution formula. Leave any integral as it is.

b) (10') Let $\vec{x}^{(1)}, \vec{x}^{(2)}, \vec{x}^{(3)}$ be three solutions that satisfy the following initial conditions,

$$\vec{x}^{(1)}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{x}^{(2)} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad \vec{x}^{(3)} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}.$$

Prove $\vec{x}^{(1)}(t) + \vec{x}^{(3)}(t) = 2\vec{x}^{(2)}(t)$.

——— *Problem 9 on the other side.* ———

9. (extra 10') The dynamics of a spring with damping effects is modeled by

$$y'' + py' + y = 0 \quad \text{with} \quad p = p(t, y, y') > 0.$$

Show that the total energy of the spring $E(t) = [y'(t)]^2 + [y(t)]^2$ is indeed non-increasing as time goes on.