Math 216 Midterm: **Answer**

[6–8 pm, July 18, 2008]

No calculators or references

Part I (28')	Part II (54')	Part III (38')	Total Score (120')

Notation: in this test, the independent variable is always t.

Part I. True or False. (4'×7)

Indicate the truthfulness of the following statements. **Correct the false ones**.

1. True. The following differential equation is linear and separable

$$\frac{dx}{dt} = -2\sin(t^2)x.$$

2. False. If we know n linearly indepedent solutions of an n-th order linear homogeneous equation, $\{y_1, y_2, ..., y_n\}$, then the general solution can always be written as their linear combination

$$y = c_1y_1 + c_2y_2 + ... + c_ny_n.$$

3. True. Let y_1, y_2 both satisfiy the following linear equation

$$L[y] = f(t).$$

Then, function $z = y_1 - y_2$ satisfies

$$L[z] = 0.$$

4. False. The Runge-Kutta method is of 4-th order when applied to differential equatios. In other words, with step size h, the cumulative error e is bounded as $|e| \leq \frac{h}{4}$. $|e| \leq Ch^4$.

5. False. The autonomous differential equation

$$\frac{dy}{dt} = y^3(y-3)$$

has a stable unstable equilibrium solution $y \equiv 3$ and an unstable stable equilibrium solution $y \equiv 0$.

6. True. The solution to equation

$$(t+1)\frac{dy}{dt} = \sin(t) + \frac{1}{t-2},\ y(0) = -2$$

exists and is unique for $t \in (-1, 2)$.

7. False. Consider a second order differential equation

$$x'' + \cos(t)x = 0$$

and 2 solutions $x_1(t)$, $x_2(t)$. Then x_1, x_2 are linearly independent if and only if $x_1(t)x_2(t) - x_1'(t)x_2'(t) \neq 0$ $x_1(t)x_2'(t) - x_2(t)x_1'(t) \neq 0$ for all $t \in (-\infty, \infty)$.

Part II. Solving equations. (54')

Find solutions to the following equations/systems. If initial conditions are given, find the specific solutions as well.

1. (10') Solve for $y = y(t)$ in

$$ty' - 2y + 3t^2 = 0,\ y(1) = 2.$$

Divide the eq. with t

$$y' - \frac{2}{t}y + 3t = 0.$$

Multiply with the integrating factor $\phi(t) = e^{\int \frac{-2}{t}dt} = e^{-2\ln|t|} = t^{-2}$,

$$\frac{d}{dt}\left(t^{-2}y\right) = -\frac{3}{t}.$$

$$t^{-2}y = \int -\frac{3}{t} = -3\ln|t| + C.$$

Plug in the initial condition to find that $C = 2$.

2. (10') Solve for $x = x(t)$ in

$$e^t x' + x + x^{-1} = 0.$$

Divide the eq with e^t,

$$\frac{dx}{dt} = -\frac{x + x^{-1}}{e^t}.$$

It is a separable eq,

$$\frac{dx}{x + x^{-1}} = -e^{-t} dt.$$

Integrate LHS,

$$\int \frac{dx}{x + x^{-1}} = \int \frac{x}{x^2+1} dx \overset{u=x^2+1}{=\!=\!=} \int \frac{1}{2u} du = \frac{1}{2} \ln |u| = \frac{1}{2} \ln(x^2+1).$$

The integral of the RHS is obviously e^{-t}. Therefore the solution

$$\frac{1}{2} \ln(x^2+1) = e^{-t} + C.$$

3. (12') Solve for $x = x(t)$, $y = y(t)$ in

$$x' = -3x + y \tag{1}$$

$$y' = -2x \tag{2}$$

Apply $\frac{d}{dt}$ to eq (1) and add it to eq (2),

$$x'' + 3x' + 2x = 0$$

Charateristic equation $\lambda^2 + 3\lambda + 2 = 0$ yields two distinct real eigenvalues $\lambda = -1, -2$. Thus

$$x(t) = C_1 e^{-t} + C_2 e^{-2t}.$$

Plug it into equation (1), we have

$$y = x' + 3x = 2C_1 e^{-t} + C_2 e^{-2t}.$$

Or, alternatively, from equation (2), we have $x = -y'/2$. Plug it into equation (1),

$$-\frac{y''}{2} = \frac{3}{2} y' + y$$

which yields

$$y = D_1 e^{-t} + D_2 e^{-2t}.$$

Then plug it into eq (2), we get

$$x = \frac{D_1}{2} e^{-t} + D_2 e^{-2t}.$$

4. (12') Solve for $y = y(t)$ in

$$y'' - y = 2t - e^t.$$

The homogeneous solution is

$$y_h = C_1 e^t + C_2 e^{-t}.$$

Assume a particular solution as

$$y_p = At + B + Dte^t.$$

Then $y_p'' = 2De^t + Dte^t$. Plug y_p, y_p'' into the equation

$$2De^t + Dte^t - (At + B + Dte^t) = 2t - e^t.$$

Comparing coefficients, we get $A = -2, B = 0, D = -1/2$. So the answer is

$$y = C_1 e^t + C_2 e^{-t} - 2t - \frac{1}{2} te^t.$$

5. (10') Use the improved Euler's method on equation

$$\frac{dx}{dt} + x^2 = t^2 - t,\ x(0) = 1.$$

With step size $h = 1$, fill out the following table. Show your computation.

The equation can be rearranged as $\frac{dx}{dt} = t^2 - t - x^2$ so we let

$$f(t, x) = t^2 - t - x^2.$$

Since $\tilde{x}_1$ is already predicted in the table, by the trapezoidal rule,

$$x_1 = x_0 + \frac{1}{2}\left(f(t_0, x_0) + f(t_1, \tilde{x}_1)\right) = \frac{1}{2}.$$

Next, predict $\tilde{x}_2$ using Euler's method

$$\tilde{x}_2 = x_1 + 1 \cdot f(t_1, x_1) = \frac{1}{4}.$$

And x_2 is already given in the table.

n	t_n	x_n	predicted $\tilde{x}_n$
0	0	1	(empty)
1	1	1/2	0
2	2	43/32	1/4

Part III. Applications. (38')

1. (14') An object with mass $m = 5$ is released from height H with initial velocity $v(0) = 0$. It encounters air resistance given by $-10v$ where v is the velocity. Assume the gravity constant $g = 10$.

   (a) What is the limiting velocity $\lim_{t\to\infty} v(t)$?
   The DE in terms of v is $m\frac{dv}{dt} = -10v - mg$, i.e.,

   $$5\frac{dv}{dt} = -10v - 10 \cdot 5.$$

   Set the RHS zero and find the equilibrium solution, i.e. the limiting velocity

   $$v^{limit} = -5.$$

   (b) What is the velocity and height of the object at time t?
   The DE is both linear and separable. By either method, we get

   $$v(t) = -5 + Ce^{-2t}.$$

   Plug in the initial condition $v(0) = 0$ to get $C = 5$. The the height then equals the sum of initial height H and the distance it travels,

   $$h(t) = H + \int_0^t v(s)\,ds = H + \int_0^t (-5 + 5e^{-2s})\,ds = H - 5t - \frac{5}{2}\left(e^{-2t} - 1\right).$$

2. (16') Consider a spring-mass system with mass $m = 2$ and spring constant $k = 1$.

   (a) If the motion is undamped, what kind of external force $F(t)$ can cause resonance in the system? Give an example of such $F(t)$ and briefly justify your answer.

Resonance occurs if $F(t)$ contains a frequency component equal or close to the natural frequency of the system, that is, $\sqrt{k/m} = \sqrt{1/2}$. For instance, if, $F(t) = \cos(t/\sqrt{2})$, then the solution is given by

$$x(t) = C_1 \sin(t/\sqrt{2}) + C_2 \cos(t/\sqrt{2}) + t(C_3 \sin(t/\sqrt{2}) + C_4 \cos(t/\sqrt{2}))$$

which exhibits growing oscillations with amplitude proportional to t.

(b) Let's consider damping constant c. Assume the external force $F(t) \equiv 0$. Find the value of c such that the pseudo-frequency of the damped sytem equals half of the natural frequence of the undamped system.
Natural frequency, as shown above, is $\omega_0 = \sqrt{1/2}$.
Pseudo-frequency comes from the imaginary part of the eigenvalues

$$\lambda = \frac{-2c \pm \sqrt{4mk - c^2}}{2m}.$$

Therefore, $\omega = \sqrt{8 - c^2}/4$. Set it equal to $\omega_0/2$ and solve to get $c = \sqrt{6}$.

(c) Suppose $c \in (0, 2\sqrt{2})$. Show that the pseudo-frequency is always less than the natural frequence.
Since $c \in (0, 2\sqrt{2})$, the pseudo-frequency, as show above, satisfies

$$\omega = \frac{\sqrt{8 - c^2}}{4} < \frac{\sqrt{8 - 0^2}}{4} = \sqrt{\frac{1}{2}} = \omega_0.$$

3. (8') Suppose the population $P(t)$ of certain species is governed by

$$\frac{dP}{dt} = k_1 P \left(M - e^{k_2 P}\right), \; P(0) > 0.$$

Here, k_1, k_2, M are some positive parameters. Show that

$$\lim_{t \to \infty} P(t) = \frac{1}{k_2} \ln(M).$$

Setting $\frac{dP}{dt} = 0$ we get two equilibrium solutions

$$P_1 \equiv 0, \; P_2 \equiv \frac{1}{k_2} \ln(M).$$

By phase diagram analysis, $P' < 0$ for $P \in (P_2, \infty)$ and $P' > 0$ for $P \in (P_1, P_2)$. Thus, P_2 is a stable equilibrium, which means, $\lim_{t \to \infty} P(t) = P_2$ if $P(0) > 0$.