Math 316 Final Exam

Name ____________

December 18th, 2007 1:30–3:30pm

No Calculators. No textbooks. No references.

Problems on **both sides**.

****

Write by hand and sign the following **honor pledge**:

I pledge on my honor that I have not given or received any unauthorized assistance on this examination.

Signature:______________

****

Total score:______________

Problem	1 & 2	3 & 4	5 & 6	7 & 8	9 & 10
Score					

Table of Laplace Transform

$$\mathcal{L}\{1\} = \frac{1}{s} \tag{1}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \qquad n \text{ is positive integer} \tag{2}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s+a} \tag{3}$$

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2} \tag{4}$$

$$\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2} \tag{5}$$

$$\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}\mathcal{L}\{f(t)\} \tag{6}$$

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - ... - f^{(n-1)}(0) \tag{7}$$

1. Fill in the blanks.

a) (10') The definition of Laplace transform is

$$\mathcal{L}\{f(t)\} = \underline{\hspace{4cm}}$$

b) (10') Let y_1, y_2, y_3 to be 3 solutions to a **third** order linear equation

$$y''' + py'' + qy' + ry = 0.$$

The definition of the Wroskian $W[y_1, y_2, y_3]$ is,

$$\underline{\hspace{10cm}}$$

c) (10') Consider the following equation

$$2t^2x'' + tx' - 3x = 0, \qquad t > 0.$$

Knowing one particular solution is $x_1(t) = t^{-1}$, we can postulate that another linearly independent solution is of the form ____________________. **Do not solve the equation.**

2. (30') Consider the following equation,

$$\frac{d^2y}{dx^2} + (1 + 2x^2)y = 0, \quad y(0) = -1, \, y'(0) = 2.$$

Fill in the first several coefficients in the following series solution. **Show all your work.**

$$y(x) = -1 + \underline{\hspace{1cm}}x + \frac{1}{2}x^2 + \underline{\hspace{1cm}}x^3 + ...$$

3. (30') Find inverse Laplace transform

$$\mathcal{L}^{-1}\left(\frac{9}{s^2(s+3)}\right).$$

4. A spring-mass system is modeled as

$$mx'' + 4x + 2x = 0.$$

Here $x = x(t)$ denotes the displacement and m denotes the mass.

a) (15') Find the critical mass m^{cr} at which the dynamics of the system changes from oscillatory to non-oscillatory.

b) (15') Let $m = m^{cr}$ as found in part a). Find the general solution.

5. a) (10') A linear system with constant coefficients $\vec{x}' = A\vec{x}$ has the origin as an

asymptotically stable critical point if ________________________________ are negative.

b) (20') Consider a linear system with constant coefficients $\vec{x}' = A\vec{x}$ such that

$$e^{At} = \begin{pmatrix} e^t & 0 \\ 2e^t - 2e^{3t} & e^{3t} \end{pmatrix}.$$

Find the solution $\vec{x}(t)$ with initial condition $\vec{x}(0) = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

6. a) (15') Find a particular solution to the equation

$$x'' + 2x = e^{-t}.$$

Hint: Use the method of undetermined coefficient.

b) (15') Combining part a) and the fact that $x = 4t$ is a particular solution to the equation $x'' + 2x = 8t$, find the **general** solution to the equation

$$x'' + 2x = 8t + e^{-t}.$$

7. (30') The population dynamics of two competing species can be described by the system

$$\begin{cases} x' = x(1 - x - y) \\ y' = y(2 - x - 5y) \end{cases}$$

Discuss the stability of $(x_0, y_0) = \left(\frac{3}{4}, \frac{1}{4} \right)$ as a critical point. DO NOT discuss other critical points.

8. (30') Consider the following equation

$$x'' + x' - 2x = u_3(t)\sin(t - 3), \qquad x(0) = 0, x'(0) = -2.$$

Here $u_3(t) = \begin{cases} 0, & t < 3 \\ 1, & t \geq 3 \end{cases}$. Find the Laplace transform of the solution $\mathcal{L}\{x\}$. **Do not solve the equation.**

9. (30') Consider a nonlinear system

$$\begin{cases} x' = -x^3 + y^5 \\ y' = -3xy^2 \end{cases}$$

With the help of Lyapunov function $V = ax^2 + by^4$ (a, b are constants yet to be determined), show that $(0, 0)$ is a stable critical point.

10. (30') Let $M(t) = \begin{pmatrix} e^t & -3e^{-10t} \\ 0 & e^{-10t} \end{pmatrix}$ to be a fundamental matrix to the **homogeneous** system $\vec{x}' = A\vec{x}$. Find the general solution to the **nonhomogeneous** system

$$\vec{x}' = A\vec{x} + \begin{pmatrix} 2e^{3t} \\ 0 \end{pmatrix}.$$

You may use the fact that $M^{-1}(t) = \begin{pmatrix} e^{-t} & 3e^{-t} \\ 0 & e^{10t} \end{pmatrix}$.

... THE END ...

... BLANK ...