MAT274 HW9

DUE IN CLASS, FRIDAY NOVEMBER 29, 2010.
READINGS: §6.1—6.4 OF EDWARDS & PENNEY

1. Given an $n \times n$ nonhomogenous system $\vec{x}' = A\vec{x} + \vec{g}$ with constant coefficients, that is, both A and $\vec{g}$ are constant. Assume A is invertible.

   i) Show that $\vec{x}^{(eq)}(t) \equiv -A^{-1}\vec{g}$ is an equilibrium solution.

   Proof. Plug $\vec{x}^{(eq)}(t) \equiv -A^{-1}\vec{g}$ into the original system

$$\begin{aligned} LHS &= 0 \qquad \text{since } -A^{-1}\vec{g} \text{ is constant} \\ RHS &= A(-A^{-1}\vec{g}) + \vec{g} = -\vec{g} + \vec{g} = 0 \end{aligned}$$

   Thus, $LHS = RHS$.

   ii) Suppose A has n distinct eigenvalues that are all negative. Show that $-A^{-1}\vec{g}$ is an asymptotically stable equilibrium by proving

$$\lim_{t\to+\infty} \vec{x}(t) = -A^{-1}\vec{g}\,, \text{ for any solution } \vec{x}(t).$$

   Hint: Look for a system satisfied by $\vec{x}(t) + A^{-1}\vec{g}$.

   Proof. Define a new function $\vec{z}(t) = \vec{x}(t) + A^{-1}\vec{g}$ and look for a new system **solely** in terms of $\vec{z}(t)$. Indeed, differentiate

$$\begin{aligned} \vec{z}\,'(t) &= \vec{x}\,'(t) \qquad \text{by definition of } \vec{x} \\ &= A\vec{x} + \vec{g} \qquad \text{by the original system} \\ &= A(\vec{z} - A^{-1}\vec{g}) + \vec{g} \qquad \text{by the definition of } \vec{z} \\ &= A\vec{z}. \end{aligned}$$

   Thus, the new function $\vec{z}(t)$ satisfies

$$z' = Az$$

   where the coefficient matrix A has n distinct negative eigenvalues. So, we can formally write down a formula for the general solution

$$z(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + ... + c_n e^{\lambda_n t}\vec{v}_n \tag{1}$$

where $(\lambda_k, \vec{v}_k)$ are n eigenpairs. The information provided in the problem states the real part of λ_k is negative and therefore, regardless of the imaginary part of λ_k,

$$\lim_{t\to+\infty} e^{\lambda_k t} = 0.$$

And, since $\vec{v}_k$ is a constant vector, regardless of its actual value, we have

$$\lim_{t\to+\infty} e^{\lambda_k t}\vec{v}_k = 0$$

Plugging it into (1), we prove that

$$\lim_{t\to+\infty} \vec{z}(t) = \vec{0}$$

and in terms of $\vec{x}(t) = \vec{z}(t) - A^{-1}\vec{g}$,

$$\lim_{t\to+\infty} \vec{x}(t) = -A^{-1}\vec{g}.$$

2. The motion of an undamped pendulum is modeled by the **nonlinear system**

$$\begin{cases} x' = y \\ y' = -sin(x) \end{cases}$$

where $x(t)$ stands for the angular displacement of the pendulum.

a) What is the physical meaning of $y(t)$?

Solution. Angular velocity.

b) List all equilibria within the rectangle $(x, y) \in [-10, 10] \times [-2, 2]$.

Solution. Set the x derivative zero in the system, we obtain the only possible equilibrium value for y is

$$y \equiv 0$$

Set the y derivative zero, we have $\sin x = 0$ and thus $x = k\pi$ where k is an integer. In the range $x \in [-10, 10]$, we have -3, -2, -1, 0, 1, 2, 3 as admissible values for k. So the equilibria in the range $(x, y) \in [-10, 10] \times [-2, 2]$ are

$$(k\pi, 0), \quad k = -3, -2, -1, 0, 1, 2, 3$$

c) Find an equation $H(x, y) = \textit{constant}$ satisfied by the trajectory $(x(t), y(t))$.

Solution. The original system can be written as

$$\begin{cases} dx/dt = y \\ dy/dt = -sin(x) \end{cases}$$

Divide them,

$$\frac{dx}{dy} = -\frac{y}{\sin x}$$

and rearrange so that x and y are separated on two sides of the equation

$$\sin x dx = -y dy$$

Upon integration

$$-\cos x = -y^2/2 + C$$

and thus $H(x, y) = y^2/2 - \cos x$.

Note that, if one wants to verify any solution trajectory $(x(t), y(t))$ satisfies $H(x(t), y(t)) \equiv$constant, it suffices to show

$$\frac{d}{dt}H(x(t), y(t)) \equiv 0 \quad \text{namely} \quad \frac{d}{dt}(y^2(t)/2 - \cos x(t)) \equiv 0.$$

For example,

$$\frac{d}{dt}\cos x(t) = -\sin x(t)\frac{d}{dt}x(t) = -\sin x(t) \cdot y(t)$$

by the chain rule and by the1st equation of the original system. Same treatment for $\frac{d}{dt}\frac{y^2(t)}{2}$.

3. Consider the dynamics of a spring-mass system modeled by

$$x'' + \gamma(x, x')x' + k(x)x = 0.$$

Here $k = k(x)$ and $\gamma = \gamma(x, x')$ are no longer constant but depend on the displacement x and velocity x'. Their second derivatives are defined and continuous.

By transforming the equation into a first order system and apply the stability theory of almost linear systems, show that $x \equiv x' \equiv 0$ is a stable critical point as long as $k(0) > 0$ and $\gamma(0, 0) > 0$.

Proof. To transform the original 2nd order equation into 1st order system, we introduce

$$y(t) := x'(t)$$

Then, establish 1st order equations with LHS being $x'(t)$ and $y'(t)$ respectively. Indeed, for x' it is easy to have

$$x'(t) = y(t) \tag{2}$$

by definition. For y', we use the second equation of the original system to establish

$$y'(t) = x''(t) = -\gamma(x, x')x' - k(x)x.$$

Notice, however, the new system has to be in terms of x, y and their derivatives **only**; therefore on the RHS of the above equation, we need to replace any occurrence of x' with its new name y

$$y'(t) = -\gamma(x, y)y - k(x)x$$

Combining it with (2), we obtain the desired 1st order system

$$\begin{cases} x' = y \\ y' = -\gamma(x, y)y - k(x)x \end{cases}$$

On to next step: linearization. A useful tool here is the Jacobian matrix that consists of all the derivative of the RHS of the above system.

$$J(x, y) = \begin{pmatrix} \partial_x y, & \partial_y y \\ \partial_x(-\gamma(x, y)y - k(x)x), & \partial_y(-\gamma(x, y)y - k(x)x) \end{pmatrix}$$

Carefully evaluate the second row of J by using the product rule

$$\partial_x(-\gamma(x, y)y - k(x)x) = -\gamma_x(x, y)y - k'(x)x - k(x)$$

$$\partial_y(-\gamma(x, y)y - k(x)x) = -\gamma_y(x, y)y - \gamma(x, y)$$

Now, since we want to study the stability of the equilibrium $x \equiv x' \equiv 0$ which amounts to $x \equiv y \equiv 0$ in terms of the new system, we set $x \equiv y \equiv 0$ in the Jacobian matrix J and simplify as much as possible

$$J(0, 0) = \begin{pmatrix} 0, & 1 \\ -k(0), & -\gamma(0, 0) \end{pmatrix}$$

Note it is important to set x, y to be some numerical values before calculating eigenvalues. The above matrix is constant. Though we don't know the exact values of $k(0)$, $\gamma(0)$, the problem does state that they are both positive

$$k(0) > 0, \qquad \gamma(0,0) > 0.$$

Now, solve for eigenvalues in

$$\det \begin{pmatrix} \lambda, & -1 \\ k(0), & \lambda + \gamma(0,0) \end{pmatrix} = 0$$

namely

$$\lambda^2 + a\lambda + b = 0$$

where we let $a = \gamma(0,0) > 0$ and $b = k(0) > 0$. The roots are

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

Now, there are two cases depending on the sign of the discriminant $a^2 - 4b$

• Case 1, $a^2 - 4b < 0$. Then we have two imaginary eigenvalues with the same real part $-a/2$ which is negative. This is all that matters in terms of stability. So, (0,0) is a spiral sink, which is stable.

• Case 1, $a^2 - 4b > 0$. Then we have two real roots

$$\lambda_1 = \frac{-a - \sqrt{a^2 - 4b}}{2}, \quad \lambda_2 = \frac{-a + \sqrt{a^2 - 4b}}{2}$$

Obviously $\lambda_1 < 0$ since $a > 0$. For λ_2, we notice that $b > 0$ and therefore $a^2 - 4b < a^2$ and upon taking square root $\sqrt{a^2 - 4b} < a$. Therefore, $-a + \sqrt{a^2 - 4b} < 0$ and we proved $\lambda_2 < 0$. Therefore, $(0,0)$ is a nodal sink, which is also stable.

4. The population of two competing species follows the system

$$\begin{cases} x' = x(1 - x - y) \\ y' = y(2 - x - 5y) \end{cases}$$

Find all the critical points of this system and indicate their types. Then plot a phase portrait and discuss whether or not these two species can coexist.

Solution. To find critical points, a.k.a., equilibria, we set $x' = 0$ in the first equation so that $x(1-x-y) = 0$, which implies

$$\text{either } x = 0 \text{ or } 1 - x - y = 0. \tag{3}$$

Likewise, set $y' = 0$ in the second equation so that $y(2-x-5y) = 0$, which leads to

$$\text{either } y = 0 \text{ or } 2 - x - 5y = 0.$$

Combined with (3), this gives us 4 cases.

• Case 1. $x = 0$, $y = 0$. We create the Jacobian matrix by differentiating the RHS of the original system w.r.t. x and y

$$J(x,y) = \begin{pmatrix} \partial_x(x(1-x-y)), & \partial_y(x(1-x-y)) \\ \partial_x(y(2-x-5y)), & \partial_y(y(2-x-5y)) \end{pmatrix}$$

and upon simplification

$$J(x,y) = \begin{pmatrix} 1-2x, & -x \\ -y, & 2-10y \end{pmatrix} \tag{4}$$

Thus, for the equilibrium $(x,y) = 0$,we have

$$J(0,0) = \begin{pmatrix} 1, & 0 \\ 0, & 2 \end{pmatrix}$$

whose eigenvalues are $1,\ \ 2$. So this is a nodal source.

• Case 2. $x = 0$ and $2 - x - 5y = 0$ which can be solved to yield $(x,y) = (0, 0.4)$. Use (4) to obtain

$$J(0,0.4) = \begin{pmatrix} 1, & 0 \\ -0.4, & -2 \end{pmatrix}$$

whose eigenvalues are $-1,\ \ 2$. So this is a saddle point.

• Case 3. $1 - x - y = 0$ and $y = 0$ which can be solved to yield $(x,y) = (1,0)$. Use (4) to obtain

$$J(0,0.4) = \begin{pmatrix} -1, & -1 \\ 0, & 2 \end{pmatrix}$$

whose eigenvalues are $1, \ -2$. So this is a saddle point.

• Case 4. $1 - x - y = 0$ and $2 - x - 5y = 0$ which can be solved to yield $(x, y) = (0.75, 0.25)$. Use (4) to obtain

$$J(0, 0.4) = \begin{pmatrix} -0.5, & -0.75 \\ -0.25, & -0.5 \end{pmatrix}$$

whose eigenvalues are $-0.5 \pm 0.25\sqrt{3}$ which are both negative. So this is a nodal sink.

The phase portrait, based on discussion above, resembles Section 6.3, Figure 6.3.13 on page 407 of the textbook. The 2 species can coexist and the asymptotic population will converge to the nodal sink at $(x, y) = (0.75, 0.25)$.