MAT274 HW9

DUE IN CLASS, FRIDAY NOVEMBER 29, 2010.
READINGS: §6.1—6.4 OF EDWARDS & PENNEY

1. Given an $n \times n$ nonhomogenous system $\vec{x}' = A\vec{x} + \vec{g}$ with constant coefficients, that is, both A and $\vec{g}$ are constant. Assume A is invertible.

   i) Show that $\vec{x}^{(eq)}(t) \equiv -A^{-1}\vec{g}$ is an equilibrium solution.

   ii) Suppose A has n distinct eigenvalues that are all negative. Show that $-A^{-1}\vec{g}$ is an asymptotically stable equilibrium by proving

   $$\lim_{t\to+\infty} \vec{x}(t) = -A^{-1}\vec{g}\,, \text{ for any solution } \vec{x}(t).$$

   Hint: Look for a system satisfied by $\vec{x}(t) + A^{-1}\vec{g}$.

2. The motion of an undamped pendulum is modeled by the **nonlinear system**

   $$\begin{cases} x' = y \\ y' = -sin(x) \end{cases}$$

   where $x(t)$ stands for the angular displacement of the pendulum.

   a) What is the physical meaning of $y(t)$?

   b) List all equilibria within the rectangle $(x, y) \in [-10, 10] \times [-2, 2]$.

   c) Find an equation $H(x, y) = \textit{constant}$ satisfied by the trajectory $(x(t), y(t))$.

3. Consider the dynamics of a spring-mass system modeled by

   $$x'' + \gamma(x, x')x' + k(x)x = 0.$$

   Here $k = k(x)$ and $\gamma = \gamma(x, x')$ are no longer constant but depend on the displacement x and velocity x'. Their second derivatives are defined and continuous.

   By transforming the equation into a first order system and apply the stability theory of almost linear systems, show that $x \equiv x' \equiv 0$ is a stable critical point as long as $k(0) > 0$ and $\gamma(0, 0) > 0$.

4. The population of two competing species follows the system

   $$\begin{cases} x' = x(1 - x - y) \\ y' = y(2 - x - 5y) \end{cases}$$

   Find all the critical points of this system and indicate their types. Then plot a phase portrait and discuss whether or not these two species can coexist.