MAT 274 HW 8

Solutions.

1. (15') Convert the following 3rd order, constant coefficient, DE

$$ax'''(t) + bx''(t) + cx'(t) + dx(t) = 0 \tag{1}$$

into an equivalent 1st order system

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}$$

and then prove that

$$a \det(\lambda I - A) = a\lambda^3 + b\lambda^2 + c\lambda + d$$

which is exactly the same as the characteristic equation of (1).

Solution. Define new unknowns

$$x_1 = x, \quad x_2 = x', \quad x_3 = x''$$

Then, obviously

$$\frac{d}{dt}x = x' \implies \frac{d}{dt}x_1 = x_2$$

$$\frac{d}{dt}x' = x'' \implies \frac{d}{dt}x_2 = x_3$$

To establish the equation with LHS being $\frac{d}{dt}x_3 = x'''$, we transform equation (1) into

$$x''' = -\frac{b}{a}x'' - \frac{c}{a}x' - \frac{d}{a}x$$

In terms of the new unknowns, this is

$$\frac{d}{dt}x_3 = -\frac{b}{a}x_3 - \frac{c}{a}x_2 - \frac{d}{a}x_1$$

Put the 3 equations together with the LHS being the $\frac{d}{dt}$ of x_1, x_2, x_3 respectively

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{d}{a} & -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

To calculate $\det(\lambda I - A)$, we write

$$\lambda I - A = \begin{pmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ \frac{d}{a} & \frac{c}{a} & \lambda + \frac{b}{a} \end{pmatrix}.$$

Its determinant consists of 6 products, many of which are zero

$$\det(\lambda I - A) = \lambda^2(\lambda + \frac{b}{a}) + (-1)^2\frac{d}{a} - \lambda(-1)\frac{c}{a} = \frac{1}{a}(a\lambda^3 + b\lambda^2 + c\lambda + d)$$

2. (3×10') **Note**: this is a long problem but you should expect the least amount of calculations.

   The following equation models a string-mass system oscillating with damping and external force,

$$u'' + 2u' + 3u = g(t), \quad u(1) = 0.5, \quad u'(1) = 1 \tag{2}$$

   (a) Rewrite the given second order equation (2) as its equivalent system of first order equations. Note: use v to represent the velocity function, i.e. $v = u'(t)$ and write the system in matrix-vector form in term of the unknown vector $\vec{x} = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$,

$$\frac{d}{dt}\vec{x} = A\vec{x} + \vec{f}, \quad \text{with initial data} \quad \vec{x}(1) = \underline{\begin{pmatrix} 0.5 \\ 1 \end{pmatrix}}. \tag{3}$$

   The coefficient matrix $A = \underline{\begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix}}$ and the source term $\vec{f} = \underline{\begin{pmatrix} 0 \\ g(t) \end{pmatrix}}$.

   Solution. Apparently the definition $v = u'(t)$ is the same as

$$\frac{d}{dt}u = v.$$

So it remains to establish an equation with LHS $\frac{d}{dt}v$ which is u''. By (2)

$$u'' = -2u' - 3u + g(t) \implies \frac{d}{dt}v = -2v - 3u + g(t).$$

Put both together

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix}$$

The blanks are filled above.

(b) Knowing $u_1 = e^{-t}\cos\sqrt{2}t$ and $u_2(t) = e^{-t}\sin\sqrt{2}t$ form a pair of linearly independent solution to the **homogeneous** equation

$$u_h'' + 2u_h' + 3u_h = 0,$$

write a fundamental matrix $M(t)$ of the 1st order **homogeneous** system

$$\frac{d}{dt}\vec{x}_h = A\vec{x}_h$$

where A is the same coefficient matrix as in part (a). *Hint:* no need to solve for any eigenvalue because we have enough information here!

Solution. By defintion, $v = u'$ and the new unknown vector

$$\vec{x} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ \frac{d}{dt}u \end{pmatrix}$$

So, two vector solutions associated with u_1, u_2 are

$$\vec{x}_1 = \begin{pmatrix} u_1 \\ \frac{d}{dt}u_1 \end{pmatrix}, \qquad \vec{x}_2 = \begin{pmatrix} u_2 \\ \frac{d}{dt}u_2 \end{pmatrix}.$$

Plug in the information about u_1, u_2 given in the problem,

$$\vec{x}_1 = \begin{pmatrix} e^{-t}\cos\sqrt{2}t \\ -e^{-t}\cos\sqrt{2}t - \sqrt{2}e^{-t}\sin\sqrt{2}t \end{pmatrix},$$

$$\vec{x}_2 = \begin{pmatrix} e^{-t}\sin\sqrt{2}t \\ -e^{-t}\sin\sqrt{2}t + \sqrt{2}e^{-t}\cos\sqrt{2}t \end{pmatrix},$$

are the fundamental matrix is just $M = (\vec{x}_1, \vec{x}_2)$. We known $\det M \neq 0$ because the Wroskian of the 2nd order equation and the 1st order system are the same, and therefore linear independence of u_1, u_2 is equivalent to linear independence of $\vec{x}_1$, $\vec{x}_2$.

(c) With the fundamental matrix $M(t)$ obtained in part (b), write down a solution formula to the system (3)

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \underline{\begin{pmatrix} e^{-t}\cos\sqrt{2}t, & e^{-t}\sin\sqrt{2}t \\ -e^{-t}\cos\sqrt{2}t - \sqrt{2}e^{-t}\sin\sqrt{2}t, & -e^{-t}\sin\sqrt{2}t + \sqrt{2}e^{-t}\cos\sqrt{2}t \end{pmatrix}}$$

$$\left(\int_{\underline{1}}^{\underline{t}} \underline{\begin{pmatrix} e^{-s}\cos\sqrt{2}s, & e^{-s}\sin\sqrt{2}s \\ -e^{-s}\cos\sqrt{2}s - \sqrt{2}e^{-s}\sin\sqrt{2}s, & -e^{-s}\sin\sqrt{2}s + \sqrt{2}e^{-s}\cos\sqrt{2}s \end{pmatrix}}^{-1} \begin{pmatrix} 0 \\ g(s) \end{pmatrix} ds + \right.$$

$$\left. \underline{\begin{pmatrix} e^{-1}\cos\sqrt{2}, & e^{-1}\sin\sqrt{2} \\ -e^{-1}\cos\sqrt{2} - \sqrt{2}e^{-1}\sin\sqrt{2}, & -e^{-1}\sin\sqrt{2} + \sqrt{2}e^{-1}\cos\sqrt{2} \end{pmatrix}}^{-1} \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} \right)$$

Do NOT calculate the matrix inverse or integral but do specify each entry in the matrices and vectors you fill in the blanks.

3. (2 × 15') Solve the following systems with **repeated** eigenvalues and indicate the types of the equilibrium (i.e. zero) solutions: source/sink/saddle point, node/spiral/center.

(a)

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & -9 \\ 6 & 10 & 6 \\ -3 & 0 & 7 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

Solution. To calculate

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 0 & -9 \\ 6 & 10-\lambda & 6 \\ -3 & 0 & 7-\lambda \end{pmatrix},$$

we notice that the second row only consists one nonzero entry, and thus expand the determinant w.r.t the second row

$$\det(A - \lambda I) = 0 + (10 - \lambda)\left[(1-\lambda)(7-\lambda) - (-3)(-9)\right] + 0$$

$$= (10-\lambda)(\lambda^2 - 8\lambda - 20)$$

Solve

$$10 - \lambda = 0 \implies \lambda_1 = 10$$

and then

$$\lambda^2 - 8\lambda - 20 = 0 \implies \lambda_2 = 10 \text{ (repeated)}\ , \qquad \lambda_3 = -2 \text{ (non-repeated)}\ .$$

For here, we already see that the equilibrium is a saddle point.
For the repeated eigenvalue $\lambda_1 = \lambda_2 = 10$, the multiplicity is 2. The eigenvector(s) comes from solving

$$(A - 10I)\vec{v} = 0, \qquad \text{i.e.} \qquad \begin{pmatrix} -9 & 0 & -9 \\ 6 & 0 & 6 \\ -3 & 0 & -3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

All 3 rows are linear dependent, so we only need to solve one

$$-9a - 9c = 0 \implies a = -c \quad \text{e.g.} \quad a = 1,\ c = -1.$$

So there is only constraint among 3 unknowns a, b, c; in other words, we can find $3 - 1 = 2$ linearly independent solutions

$$\text{set the free unknown } b = 0, \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\text{set the free unknown } b = 1, \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

Eigenvalue 10 is NOT defect.

The eigenvector associated with $\lambda_3 = -2$ can be found similarly

$$A + 2I = \begin{pmatrix} 3 & 0 & -9 \\ 6 & 12 & 6 \\ -3 & 0 & 9 \end{pmatrix}$$

The 1st and 3rd row are linearly dependent, so we only use the first two rows

$$3a - 9c = 0 \implies a = 3c,$$

Plug into the second equation

$$6a + 12b + 6c = 0 \implies 6(3c) + 12b + 6c = 0 \implies b = -2c.$$

Setting $c = 1$, we get $a = 3$, $b = -2$, and thus obtain the third eigenvector

$$\vec{v}_3 = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}.$$

Answer, the general solution is

$$\vec{x}(t) = e^{10t}\left(C_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right) + C_3 e^{-2t} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

(b)

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

Solution. The characteristic equation

$$\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 4 \\ -1 & 5-\lambda \end{pmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$$

So, $\lambda_1 = \lambda_2 = 3$ is a repeated eigenvalues. The equilibrium is a nodal source.

To calculate the eigenvector(s), we establish

$$(A-3\lambda)\vec{v}=0 \implies \begin{pmatrix}-2 & 4\\ -1 & 2\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

The two rows are linearly dependent. So, there is one constraint and 2 unknowns; thus there is one linearly independent solution, e.g.

$$\vec{v}_1=\begin{pmatrix}2\\ 1\end{pmatrix}.$$

The eigenvalue 2 is defect. To find the **generalized** eigenvector $\vec{v}_2$, we solve

$$(A-2I)\vec{v}_2=\vec{v}_1 \implies \begin{pmatrix}-2 & 4\\ -1 & 2\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}2\\ 1\end{pmatrix}$$

$$\begin{cases}-2a+4b=2\\ -a+2b=1\end{cases} \tag{4}$$

These two equations are essentially the same; so we only need solve one of them, say $-a+2b=1$. Pick $b=0$ and plug it in,we have $a=-1$. Thus, the **generalize** eigenvector

$$\vec{v}_2=\begin{pmatrix}-1\\ 0\end{pmatrix}.$$

Finally, we construct two linearly independent solution to the origianl ODE system

$$\vec{x}_1(t)=e^{2t}\vec{v}_1, \qquad \vec{x}_2(t)=e^{2t}(\vec{v}_1t+\vec{v}_2)$$

and the general solution is

$$\vec{x}(t)=C_1e^{2t}\begin{pmatrix}2\\ 1\end{pmatrix}+C_2e^{2t}\left(\begin{pmatrix}2\\ 1\end{pmatrix}t+\begin{pmatrix}-1\\ 0\end{pmatrix}\right)$$

Note that there are many other choices for $\vec{v}_2$ as long as (4) is satisfied. For example $\vec{v}_2=\begin{pmatrix}1\\ 1\end{pmatrix}$.