MAT 274 HW 8

Due Friday Nov 12 in class.

1. (15') Convert the following 3rd order, constant coefficient, DE

$$ax'''(t) + bx''(t) + cx'(t) + dx(t) = 0 \tag{1}$$

into an equivalent 1st order system

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}$$

and then prove that

$$a\det(\lambda I - A) = a\lambda^3 + b\lambda^2 + c\lambda + d$$

which is exactly the same as the characteristic equation of (1).

2. (3×10') **Note**: this is a long problem but you should expect the least amount of calculations.

The following equation models a string-mass system oscillating with damping and external force,

$$u'' + 2u' + 3u = g(t), \quad u(1) = 0.5, \quad u'(1) = 1 \tag{2}$$

(a) Rewrite the given second order equation (2) as its equivalent system of first order equations. Note: use v to represent the velocity function, i.e. $v = u'(t)$ and write the system in matrix-vector form in term of the unknown vector $\vec{x} = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$,

$$\frac{d}{dt}\vec{x} = A\vec{x} + \vec{f}, \quad \text{with initial data} \quad \vec{x}(1) = \underline{\hspace{5cm}}. \tag{3}$$

The coefficient matrix $A =$ ____________________ and the source term $\vec{g} =$ ____________________.

(b) Knowing $u_1 = e^{-t}\cos\sqrt{2}t$ and $u_2(t) = e^{-t}\sin\sqrt{2}t$ form a pair of linearly independent solution to the **homogeneous** equation

$$u_h'' + 2u_h' + 3u_h = 0,$$

write a fundamental matrix $M(t)$ of the 1st order **homogeneous** system

$$\frac{d}{dt}\vec{x}_h = A\vec{x}_h$$

where A is the same coefficient matrix as in part (a). *Hint:* no need to solve for any eigenvalue because we have enough information here!

(c) With the fundamental matrix $M(t)$ obtained in part (b), write down a solution formula to the system (3)

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \\ \_\_\_\_ \end{pmatrix} \left(\int_{\_\_}^{\_\_} \begin{pmatrix} \\ \_\_\_\_ \end{pmatrix}^{-1} \begin{pmatrix} \\ \_\_ \end{pmatrix} ds + \begin{pmatrix} \\ \_\_ \end{pmatrix}^{-1} \begin{pmatrix} 0.5 \\ 1 \end{pmatrix} \right)$$

Do NOT calculate the matrix inverse or integral but do specify each entry in the matrices and vectors you fill in the blanks.

3. (2×15') Solve the following systems with **repeated** eigenvalues and indicate the types of the equilibrium (i.e. zero) solutions: source/sink/saddle point, node/spiral/center.

(a)

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & -9 \\ 6 & 10 & 6 \\ -3 & 0 & 7 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

(b)

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$