MAT274 HW7

DUE IN CLASS, FRIDAY NOVEMBER 5, 2010.
READINGS: §5.2, 5.5, 5.6 OF EDWARDS & PENNEY

1. (10') Note that you **don't** need to completely solve the system in this problem. By calculating the eigenvalues only, prove that all solutions to

$$x'(t) = 2x - 5y, \quad y'(t) = 4x - 2y$$

are $\frac{\pi}{2}$-periodic, i.e.

$$x(t + \frac{\pi}{2}) = x(t), \quad y(t + \frac{\pi}{2}) = y(t).$$

Then, sketch a phase portrait on the $x - y$ plane. The direction of arrows can be determined by studying the signs of $dx/dt, \quad dy/dt$ at sample points, e.g. $x = 1, y = 0$ and $x = 0, y = 1$.

Solution. The coef matrix

$$A = \begin{pmatrix} 2 & -5 \\ 4 & -2 \end{pmatrix}$$

has charactersitc equation

$$\det \begin{pmatrix} 2 - \lambda & -5 \\ 4 & -2 - \lambda \end{pmatrix} = 0.$$

That is

$$\lambda^2 + 16 = 0$$

and therefore the eigenvalues are purely imaginary

$$\lambda_1 = 4i, \quad \lambda_1 = -4i.$$

Let $\vec{v}_1$ and $\vec{v}_2$ be the associated eigenvectors so that the general solution to the original system is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{4it}\vec{v}_1 + e^{-4it}\vec{v}_2. \tag{1}$$

Note that the only t-dependence of the solution is via the e^{4it} and e^{-4it} terms. Both of them are 0.5π-periodic. In fact, by the Euler's formula

$$e^{4it} = \cos(4t) + i\sin(4t), \qquad e^{-4it} = \cos(4t) - i\sin(4t)$$

and both $\cos(4t)$ and $\sin(4t)$ are 0.5π-periodic since, for example,

$$\cos(4(t+0.5\pi)) = \cos(4t+2\pi) = \cos(4t).$$

Therefore, for any solution of the form (1) must be 0.5π-periodic.

The phase portrait look like concentric ellipses that are elongated and tiltled towards the 1st and 3rd quadrants. The direction of the solution curve is counter-clockwise since, for example, at $x=1, y=0$, we have $dx/dt = 2$ and $dy/dt = 4$. For an ellipse this indicates the direction of the arrows is couternclockwise.

2. (25') Knowing the eigenpairs of the matrix

$$A = \begin{pmatrix} -2, & 5 \\ 3, & -4 \end{pmatrix}$$

are

$$\lambda_1 = -7, \ \ \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_2 = 1, \ \ \vec{v}_2 = \begin{pmatrix} 5 \\ 3 \end{pmatrix},$$

solve the nonhomogeneous system

$$\begin{cases} x'(t) = -2x + 5y + \dfrac{14}{1+e^{7t}} \\ y'(t) = 3x - 4y - \dfrac{14}{1+e^{7t}} \end{cases}$$

with initial data $x(0) = 10, \ \ y(0) = 6$. The answer is

$$\begin{cases} x(t) = 2e^{-7t}\ln(1+e^{7t}) + 10e^{t} \\ y(t) = -2e^{-7t}\ln(1+e^{7t}) + 6e^{t} \end{cases}$$

but you have to show all your work.

Solution. Note that the answer given above has typos in the coefficients of e^t.

They will be re-evaluated below.

First, the fundamental matrix associate with the *homogeneous* system is

$$M(t) = (e^{\lambda_1 t}\vec{v}_1,\ e^{\lambda_2 t}\vec{v}_2) = \begin{pmatrix} e^{-7t} & 5e^t \\ -e^{-7t} & 3e^t \end{pmatrix} \tag{2}$$

By variation of parameters, we assume the NONhomogeneous solution is

$$\vec{x}(t) = M(t)\vec{c}(t).$$

Plug into the original system $\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{f}(t)$

$$\frac{d}{dt}(M(t)\vec{c}(t)) = AM(t)\vec{c}(t) + \vec{f}(t). \tag{3}$$

Apply to the LHS the product rule and the fact that any fundamental matrix satisfies the *homogeneous* system $\frac{d}{dt}M(t) = AM(t)$,

$$\frac{d}{dt}(M(t)\vec{c}(t)) = \frac{d}{dt}M\vec{c} + M\frac{d}{dt}\vec{c} = AM\vec{c} + M\frac{d}{dt}\vec{c}.$$

Equate it with the RHS of (3) and cancell out the $AM\vec{c}$ terms

$$M(t)\frac{d}{dt}\vec{c}(t) = \vec{f}(t)$$

i.e. upon inverting $M(t)$ and integration

$$\vec{c}(t) = \int M^{-1}(t)\vec{f}(t)\,dt. \tag{4}$$

Recall the definition of $M(t)$ in (2) and calculate its inverse

$$M^{-1} = \frac{1}{\det M}\begin{pmatrix} 3e^t & -5e^t \\ e^{-7t} & e^{-7t} \end{pmatrix} = \frac{1}{8e^{-6t}}\begin{pmatrix} 3e^t & -5e^t \\ e^{-7t} & e^{-7t} \end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix} 3e^{7t} & -5e^{7t} \\ e^{-t} & e^{-t} \end{pmatrix}$$

Plug this into (4) and replace $\vec{f}(t)$ with the actual nonhomogeneous terms in the original system $\vec{f} = \begin{pmatrix} \frac{14}{1+e^{7t}} \\ -\frac{14}{1+e^{7t}} \end{pmatrix}$, we arrive at

$$\vec{c}(t) = \int \frac{1}{8}\begin{pmatrix} 3e^{7t} & -5e^{7t} \\ e^{-t} & e^{-t} \end{pmatrix}\begin{pmatrix} \frac{14}{1+e^{7t}} \\ -\frac{14}{1+e^{7t}} \end{pmatrix} dt = \int \begin{pmatrix} \frac{14e^{7t}}{1+e^{7t}} \\ 0 \end{pmatrix} dt$$

To evaluate the integral of $\frac{14}{1+e^{7t}}$, we use substitution $u = 1 + e^{7t}$ and thus $du = 7e^{7t}dt$,

$$\int \frac{14}{1+e^{7t}}\,dt = \int \frac{2}{u}du = 2\ln|u| + C = 2\ln(1+e^{7t}) + C_1.$$

Thus,

$$\vec{c}(t) = \int \begin{pmatrix} \frac{14e^{7t}}{1+e^{7t}} \\ 0 \end{pmatrix} dt = \begin{pmatrix} 2\ln(1+e^{7t}) + C_1 \\ C_2 \end{pmatrix}$$

All in all, we found the vector $\vec{c}(t)$ in the assumed form $\vec{x}(t) = M(t)\vec{c}(t)$, therefore the general solution is (using (2)

$$\vec{x}(t) = M(t)\vec{c}(t) = \begin{pmatrix} e^{-7t} & 5e^t \\ -e^{-7t} & 3e^t \end{pmatrix} \begin{pmatrix} 2\ln(1+e^{7t}) + C_1 \\ C_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{-7t}\ln(1+e^{7t}) \\ -2e^{-7t}\ln(1+e^{7t}) \end{pmatrix} + \begin{pmatrix} e^{-7t} & 5e^t \\ -e^{-7t} & 3e^t \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

Now, to find constants C_1, C_2, we plug in the initial condition $x(0) = 10, y(0) = 6$,

$$\begin{pmatrix} 10 \\ 6 \end{pmatrix} = \begin{pmatrix} 2\ln 2 \\ -2\ln 2 \end{pmatrix} + \begin{pmatrix} 1 & 5 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

Therefore

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ -1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 10 - 2\ln 2 \\ 6 + 2\ln 2 \end{pmatrix}$$

$$= \frac{1}{8}\begin{pmatrix} 3 & -5 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 10 - 2\ln 2 \\ 6 + 2\ln 2 \end{pmatrix} = \begin{pmatrix} -2\ln 2 \\ 2 \end{pmatrix}$$

and the specific solution should be

$$\vec{x}(t) = \begin{pmatrix} 2e^{-7t}\ln(1+e^{7t}) + 10e^t - (2\ln 2)e^{-7t} \\ -2e^{-7t}\ln(1+e^{7t}) + 6e^t + (2\ln 2)e^{-7t} \end{pmatrix}.$$

3. $(5 \times 5')$ Note that in this problem, we intentionally **avoid** any trigonometric functions but instead work with complex numbers directly.

Consider a 3×3 system of DE

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 2 & -2 & 0 \\ 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

Let $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$ be the unknown vector and

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 2 & -2 & 0 \\ 0 & 4 & 0 \end{pmatrix}$$

be the coeffiecient matrix.

(a) Show that the charactersitc equation is

$$\det(A - \lambda I) = -\lambda^2(\lambda + 2) + 16,$$

and then, show that the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = -2 + 2i$, $\lambda_3 = -2 - 2i$.

Solution. $\det(A-\lambda I) = \det \begin{pmatrix} -\lambda & 0 & 2 \\ 2 & -2-\lambda & 0 \\ 0 & 4 & -\lambda \end{pmatrix}$. Use your favorite method to calculate the determinant of a 3×3 matrix.

To verify the solution, we first note that 2 is an obvious root. Then, we only need to verify λ_2 is a root since λ_3 is its complex conjugate and every coefficients in the charactersitc equation $-\lambda^2(\lambda+2) + 16 = 0$ is real.

The following calculation may provide some details

$$\lambda_2^2 = (-2+2i)^2 = (-2)^2 + 2(-2)(2i) + (2i)^2 = 4 - 8i - 4 = -8i$$

Thus,

$$-\lambda_2^2(\lambda_2 + 2) + 16 = 8i((-2+2i) + 2) + 16 = 8i(2i) + 16 = 0.$$

Verified!

(b) If we know the first two associated eigenvectors are

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \; \vec{v}_2 = \begin{pmatrix} 1 \\ -i \\ -1+i \end{pmatrix},$$

what is the third one $\vec{v}_3$? No calculation is needed here.

Solution. $\vec{v}_3$ is the complex conjugate of $\vec{v}_2$,

$$\vec{v}_3 = \begin{pmatrix} 1 \\ i \\ -1-i \end{pmatrix}$$

(c) Construct a fundamental matrix $M(t)$ in terms of the information give above and **don't** convert it into real form.

Solution. $M(t) = (e^{\lambda_1 t}\vec{v}_1,\; e^{\lambda_2 t}\vec{v}_2,\; e^{\lambda_3 t}\vec{v}_3)$.

(d) Use the same fundamental matrix from above to find a solution fomular for the initial data

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0.$$

Again, do not convert it into real form. Use the fact

$$\begin{pmatrix} 2, & 1, & 1, \\ 1, & -i, & i, \\ 2, & -1+i, & -1-i \end{pmatrix}^{-1} = \begin{pmatrix} 0.2, & 0.2, & 0.2 \\ 0.3-0.1i, & -0.2+0.4i, & -0.2-0.1i \\ 0.3+0.1i, & -0.2-0.4i, & -0.2+0.1i \end{pmatrix}.$$

Solution. The general solution is

$$\vec{x}(t) = M(t)\vec{c} = \begin{pmatrix} 2e^{2t} & e^{(-2+2i)t} & e^{(-2-2i)t} \\ e^{2t} & e^{(-2+2i)t}(-i) & e^{(-2-2i)t}(i) \\ 2e^{2t} & e^{(-2+2i)t}(-1+i) & e^{(-2-2i)t}(-1-i) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}.$$

Plug in the initial condition

$$\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 2, & 1, & 1, \\ 1, & -i, & i, \\ 2, & -1+i, & -1-i \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}$$

and thus

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 2, & 1, & 1, \\ 1, & -i, & i, \\ 2, & -1+i, & -1-i \end{pmatrix}^{-1} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

...(by the inverse given in the problem) $= \begin{pmatrix} 0.2, & 0.2, & 0.2 \\ 0.3-0.1i, & -0.2+0.4i, & -0.2-0.1i \\ 0.3+0.1i, & -0.2-0.4i, & -0.2+0.1i \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$

$$= \begin{pmatrix} 0.2(x_0+y_0+z_0) \\ etc \\ etc \end{pmatrix}$$

So, the solution, in terms of the initial condition, is

(5)

$$\vec{x}(t) = M(t) \begin{pmatrix} 0.2(x_0+y_0+z_0) \\ etc \\ etc \end{pmatrix} = 0.2(x_0+y_0+z_0)e^{2t}\vec{v}_1 + (etc)e^{(-2+2i)t}\vec{v}_2 + (etc)e^{(-2-2i)t}\vec{v}_3$$

(e) Prove that, with the initial data symbolically specified as above, the asymptotic behavior of the solution satisfies

$$\lim_{t\to\infty} (\vec{x}(t) - 0.2(x_0+y_0+z_0)e^{2t}\vec{v}_1) = 0.$$

Here, $\vec{v}_1$ is the eigenvector specified in part (3b).

Solution. This is obvious if we use (5),

$$\vec{x}(t) - 0.2(x_0+y_0+z_0)e^{2t}\vec{v}_1 = (etc)e^{(-2+2i)t}\vec{v}_2 + (etc)e^{(-2-2i)t}\vec{v}_3$$

and since $\vec{v}_2$ and $\vec{v}_3$ and the "etc" terms are just constant, the $e^{(-2+2i)t} = e^{-2t}(\cos 2t + i\sin 2t)$ and $e^{(-2-2i)t} = e^{-2t}(\cos 2t - i\sin 2t)$ terms both tend to zero as $t \to +\infty$ and therefore the RHS above approaches zero as well.