MAT274 HW7

DUE IN CLASS, FRIDAY NOVEMBER 5, 2010.
READINGS: §5.2, 5.5, 5.6 OF EDWARDS & PENNEY

1. (10') Note that you **don't** need to completely solve the system in this problem. By calculating the eigenvalues only, prove that all solutions to

$$x'(t) = 2x - 5y, \quad y'(t) = 4x - 2y$$

are $\frac{\pi}{2}$-periodic, i.e.

$$x(t+\frac{\pi}{2}) = x(t), \quad y(t+\frac{\pi}{2}) = y(t).$$

Then, sketch a phase portrait on the $x-y$ plane. The direction of arrows can be determined by studying the signs of $dx/dt, \quad dy/dt$ at sample points, e.g. $x=1, y=0$ and $x=0, y=1$.

2. (25') Knowing the eigenpairs of the matrix

$$A = \begin{pmatrix} -2, & 5 \\ 3, & -4 \end{pmatrix}$$

are

$$\lambda_1 = -7, \ \ \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\lambda_2 = 1, \ \ \vec{v}_2 = \begin{pmatrix} 5 \\ 3 \end{pmatrix},$$

solve the nonhomogeneous system

$$\begin{cases} x'(t) = -2x + 5y + \dfrac{14}{1+e^{7t}} \\ y'(t) = 3x - 4y - \dfrac{14}{1+e^{7t}} \end{cases}$$

with initial data $x(0) = 10, \quad y(0) = 6$. The answer is

$$\begin{cases} x(t) = & 2e^{-7t}\ln(1+e^{7t}) + 10e^t \\ y(t) = & -2e^{-7t}\ln(1+e^{7t}) + 6e^t \end{cases}$$

but you have to show all your work.

3. $(5 \times 5')$ Note that in this problem, we intentionally **avoid** any trigonometric functions but instead work with complex numbers directly.

   Consider a 3×3 system of DE

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 2 & -2 & 0 \\ 0 & 4 & 0 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}.$$

   Let $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$ be the unknown vector and

$$A = \begin{pmatrix} 0 & 0 & 2 \\ 2 & -2 & 0 \\ 0 & 4 & 0 \end{pmatrix}$$

   be the coeffiecient matrix.

   (a) Show that the charactersitc equation is

$$\det(A - \lambda I) = -\lambda^2(\lambda + 2) + 16,$$

   and then, show that the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = -2 + 2i$, $\lambda_3 = -2 - 2i$.

   (b) If we know the first two associated eigenvectors are

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ -i \\ -1 + i \end{pmatrix},$$

   what is the third one $\vec{v}_3$? No calculation is needed here.

   (c) Construct a fundamental matrix $M(t)$ in terms of the information give above and **don't** convert it into real form.

(d) Use the same fundamental matrix from above to find a solution fomular for the initial data

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0.$$

Again, do not convert it into real form. Use the fact

$$\begin{pmatrix} 2, & 1, & 1, \\ 1, & -i, & i, \\ 2 & -1+i & -1-i \end{pmatrix}^{-1} = \begin{pmatrix} 0.2, & 0.2, & 0.2 \\ 0.3-0.1i, & -0.2+0.4i, & -0.2-0.1i \\ 0.3+0.1i, & -0.2-0.4i, & -0.2+0.1i \end{pmatrix}.$$

(e) Prove that, with the initial data symbolically specified as above, the asymptotic behavior of the solution satisfies

$$\lim_{t\to\infty} (\vec{x}(t) - 0.2(x_0 + y_0 + z_0)e^{2t}\vec{v}_1) = 0.$$

Here, $\vec{v}_1$ is the eigenvector specified in part (3b).