MAT 274 HW 6

Due Friday Oct 29 in class.
(15'+20'+20'+15')

1. Find the general solution to

$$x''(t) + x'(t) + 3x(t) + 5\sin 2t = -12e^{-3t}$$

and show that any solution $x(t)$ to this DE satisfies the following asymptotic behavior

$$\lim_{t\to\infty} (x(t) - \sin 2t - 2\cos 2t) = 0.$$

Solution. **Note**: in the first version, the equation was

$$x'' + 2x' + 3x + ...$$

which was corrected as

$$x'' + x' + 3x + ...$$

It is BOTH ok as long as the solution is faithful to the Mathematics — the old version won't lead to the desired conclusion, though.
We now use

$$x'' + x' + 3x + ...$$

The complementary solution solves

$$x_c'' + x_c' + 3x_c = 0$$

for which the char. equation is

$$r^2 + r + 3 = 0$$

The roots are complex

$$r_{1,2} = \frac{-1 \pm i\sqrt{11}}{2}$$

and thus the complimentary solution is

$$x_c = e^{\frac{-t}{2}} \left(C_1 \cos(\frac{\sqrt{11}}{2}t) + C_2 \sin(\frac{\sqrt{11}}{2}t) \right)$$

To find a particular solution, we make a guess

$$x_p = A \sin 2t + B \cos 2t + De^{-3t}$$

and plug it into the original NONhomogeneous equation

$$\frac{d^2}{dt^2} \left(A \sin 2t + B \cos 2t + De^{-3t} \right) + \frac{d}{dt} \left(A \sin 2t + B \cos 2t + De^{-3t} \right) +$$

$$3 \left(A \sin 2t + B \cos 2t + De^{-3t} \right) + 5 \sin 2t = -12e^{-3t}$$

that is

$$(-4A - 2B + 3A) \sin 2t + (-4B + 2A + 3B) \cos 2t + (9D - 3D + 3D)e^{-3t} + 5 \sin 2t = -12e^{-3t}$$

or

$$(-A - 2B + 5) \sin 2t + (2A - B) \cos 2t + (12 + 9D)e^{-3t} = 0$$

So, all coef should equal zero

$$-A - 2B + 5 = 0, \quad 2A - B = 0, \quad 12 + 9D = 0$$

from which the constants can be solved

$$A = 1, \;\; B = 2, \;\; D = -4/3$$

Thus, the particular solution is found and the general solution to the original NONhomogeneous equation is

$$x = x_p + x_c = \sin 2t + 2 \cos 2t - (4/3)e^{-3t} +$$

$$e^{\frac{-t}{2}} \left(C_1 \cos(\frac{\sqrt{11}}{2}t) + C_2 \sin(\frac{\sqrt{11}}{2}t) \right)$$

and the limit

$$\lim_{t\to\infty}(x-(\sin 2t+2\cos 2t)) =$$

$$\lim_{t\to\infty} -(4/3)e^{-3t}+e^{\frac{-t}{2}}\left(C_1\cos(\frac{\sqrt{11}}{2}t)+C_2\sin(\frac{\sqrt{11}}{2}t)\right)$$

is of course zero due to the negative exponent e^{-3t} and $e^{-t/2}$.

2. Consider the system

$$(*) \qquad \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

from the previous problem.

a) Find two linearly independent solutions that are also **REAL**;

Solution. $\det(\lambda I - A) =$

$$\det\begin{pmatrix} \lambda-2, & 1 \\ -2, & \lambda-4 \end{pmatrix} = (\lambda-2)(\lambda-4)+2$$

$$= \lambda^2 - 6\lambda + 10 = (\lambda-3)^2+1$$

So, the eigenvalues are

$$\lambda_1 = 3+i, \qquad \lambda_2 = 3-i.$$

Now, we solve

$$(\lambda_1 - A)\vec{v}_1 = 0$$

that is

$$\begin{pmatrix} 3+i-2, & 1 \\ -2, & 3+i-4 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0.$$

Since we chose λ_1 to be an eigenvalue, the rows of $\lambda_1 I - A$ are bound to be linearly DEPENDENT; thus, we only need to consider the first equation above

$$(1+i)a + b = 0 \implies \text{one solution can be } a=-1, b=1+i$$

Thus,

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 1+i \end{pmatrix} \tag{1}$$

— **Note**: any nonzero multiple of the above vector will do, say

$$\vec{v}_1 = \begin{pmatrix} -1+i \\ 2 \end{pmatrix} \tag{2}$$

Consequently, $\vec{v}_2$, as the complex conjugate of $\vec{v}_1$ in (1), is given by

$$\vec{v}_2 = \begin{pmatrix} -1 \\ 1-i \end{pmatrix}$$

The general solution is therefore

$$\vec{x}(t) = C_1 e^{(3+i)t} \begin{pmatrix} -1 \\ 1+i \end{pmatrix} + C_2 e^{(3-i)t} \begin{pmatrix} -1 \\ 1-i \end{pmatrix}. \tag{3}$$

By setting $C_1 = C_2 = 1/2$, we retrieve the <u>real component</u> of

$$e^{(3+i)t} \begin{pmatrix} -1 \\ 1+i \end{pmatrix} = e^{3t}(\cos t + i \sin t) \begin{pmatrix} -1 \\ 1+i \end{pmatrix} = \tag{4}$$

$$e^{3t} \begin{pmatrix} -(\cos t + i \sin t) \\ (\cos t + i \sin t)(1+i) \end{pmatrix} = e^{3t} \begin{pmatrix} -\cos t - i \sin t \\ (\cos t - \sin t) + i(\cos t + \sin t) \end{pmatrix}$$

namely, the first solution that is real

$$\vec{x}_1 = e^{3t} \begin{pmatrix} -\cos t \\ \cos t - \sin t \end{pmatrix}$$

By setting $C_1 = 1/(2i)$ and $C_2 = -1/(2i)$ in (3), we retrieve the <u>imaginary component</u> of (4), namely, the second solution that is real

$$\vec{x}_2 = e^{3t} \begin{pmatrix} -\sin t \\ \cos t + \sin t \end{pmatrix}$$

— **Note**: if one chose (2) instead of (1) for $\vec{v}_1$, the the two linearly independent solutions that are also real can be

$$e^{3t}\begin{pmatrix} -\cos t - \sin t \\ 2\cos t \end{pmatrix}, \quad \text{and} \quad e^{3t}\begin{pmatrix} \cos t - \sin t \\ 2\sin t \end{pmatrix} \tag{5}$$

b) Use the Wronskian to prove linear independency;

Solution.

$$W[\vec{x}_1, \vec{x}_2] = \det(\vec{x}_1, \vec{x}_2) = -e^{6t} \neq 0$$

— Note: for the alternative solutions in (5), the Wronskian is $-2e^{6t}$.

c) Draw a phase portrait **by hand** and justify your choice of source/sink, node/spiral/saddle point. In the case of spiral, also justify the clockwise/counterclockwise direction of rolling.

Solution. Spiral source. Direction of rolling can be determined by checking the signs of dx/dt, dy/dt at say $(x, y) = (1, 0)$ directly based on the original system. (turns out, $dx/dt > 0$, $dy/dt > 0$ at $(x, y) = (1, 0)$; therefore, counterclockwise rolling)

3. Consider the same system (*) as in the previous problem.

a) Find the particular solution satifying initial condition $x(0) = -2$, $y(0) = 1$;

$$\vec{x} = 2\vec{x}_1 - x_2 = e^{3t}\begin{pmatrix} -2\cos t + \sin t \\ \cos t - 3\sin t \end{pmatrix}$$

b) Show that, for this particular solution,

$$2x^2(t) + y^2(t) + 2x(t)y(t) = 5e^{6t}$$

and therefore

$$(**) \qquad \frac{d}{dt}[2x^2(t) + y^2(t) + 2x(t)y(t)] = 6[2x^2(t) + y^2(t) + 2x(t)y(t)]$$

Solution. Direct calculation. c) Prove that (**) is true for ANY solution to (*).

Solution. Apply the chain rule to the LHS of (**)

$$\frac{d}{dt}[2x^2(t) + y^2(t) + 2x(t)y(t)] = 4xx' + 2yy' + 2x'y + 2xy'$$

Replace x' and y' with the RHS of (*)

$$... = 4x(2x-y) + 2y(2x+4y) + 2(2x-y)y + 2x(2x+4y)$$

Upon simplification, this equals the RHS of (**).

4. With the help of Problem 2, find the general solution to

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 6e^{-t} \\ 4e^{-t} \end{pmatrix}$$

You may start with finding the particular solution.
Solution. Due to the simple structure of the nonhomogeneous terms in the above system, we use a "guessed" particular solution

$$\vec{x}_p = \begin{pmatrix} Ae^{-t} \\ Be^{-t} \end{pmatrix}$$

and plug it into the above system. Turns out $A = 2, \quad B = 0$. Ans is

$$\vec{x} = \begin{pmatrix} 2e^{-t} \\ 0 \end{pmatrix} + \vec{x}_c$$

where the complimentary solution $\vec{x}_c$ is copied directly from combining two linearly independent solutions obtained in 2a).