MAT274 Homework 3

Due: Friday 9/17

Readings: Chapter 3 of *Edwards & Penney*

Points: 12'+18'+23'+22'=75'

1. (12 points) Knowing that $y = x^4$ and $y = \frac{1}{x}$ are both solutions to

$$xy'' - 2y' - \frac{4y}{x} = 0,$$

prove that they are linearly independent on $x \in (0, \infty)$ and $x \in (-\infty, 0)$. Then, prove that any initial conditions of the form

$$y(x_0) = y_1, \quad y'(x_0) = y_2$$

with $x_0 \neq 0$ will guarantee a particular solution for either $x \in (0, \infty)$ or $x \in (-\infty, 0)$.

Solution. *The Wronskian is*

$$W[x^4,\ x^{-1}] = \det \begin{pmatrix} x^4 & x^{-1} \\ \frac{d}{dx}(x^4) & \frac{d}{dx}(x^{-1}) \end{pmatrix} = \det \begin{pmatrix} x^4 & x^{-1} \\ 4x^3 & -x^{-2} \end{pmatrix} = -5x^2$$

It is nonzero on any interval EXCLUDING $x = 0$. Thus, x^4 and x^{-1} are linearly independent on $(-\infty, 0)$ and on $(0, \infty)$.

Now, if initial conditions are given as in the problem, then we plug them into the general solution

$$y(x) = c_1 x^4 + c_2 x^{-1} \tag{1}$$

and its derivative

$$y'(x) = c_1 4x^3 + c_2(-x^{-2}) \tag{2}$$

Namely, set $x = x_0$ and $y = y_1$ in (1)

$$c_1 x_0^4 + c_2 x_0^{-1} = y_1$$

and set $x = x_0$ and $y' = y_2$ in (2)

$$c_1 4x_0^3 + c_2(-x_0^{-2}) = y_2.$$

The last 2 equations form a linear system for the unknown pair c_1, c_2 — again, we see linearity depends on how the unknown appears and the coef of the unknowns don't matter.

Now, one can solve for c_1, c_2 in this linear system using e.g. substituion or row reduction. But, a much more fundamental approach is to use the matrix-vector language: the unknown vector $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ satisfies the linear system

$$A\vec{c} = \vec{y}$$

where $A = \begin{pmatrix} x_0^4 & x_0^{-1} \\ 4x_0^3 & -x_0^{-2} \end{pmatrix}$ is the coefficient matrix in this linear system and $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ is the RHS vector. This system admits a unique solution

$$\vec{c} = A^{-1}\vec{y}$$

if and only if the inverse of A exists. And, by Linear Algebra, a matrix is invertible if and only if its determinant is nonzero. So, we just need to require $\det(A) = \det \begin{pmatrix} x_0^4 & x_0^{-1} \\ 4x_0^3 & -x_0^{-2} \end{pmatrix} \neq 0$, which is exactly the condition that the Wronskian $W[x_0^4, x_0^{-1}]$ is nonzero. And, this is the case for $x_0 \in (-\infty, 0)$ and $x_0 \in (0, \infty)$ as argued in the first part of the solution.

2. Consider a **nonhomogenenous** linear equation

$$xy'' - 2y' - \frac{4y}{x} = 12 \tag{3}$$

(a) (6 points) If $y = f(x)$ and $y = g(x)$ are two solutions to the above DE, prove that $y = f(x) - g(x)$ satisfies

$$xy'' - 2y' - \frac{4y}{x} = 0 \tag{4}$$

which is the **homogeneous** counterpart of the nonhomogeneous DE (3).

Solution. *Since $y = f(x)$ solves (3), we set $y = f(x)$ in (3),*

$$xf''(x) - 2f'(x) - \frac{4}{x}f(x) = 12 \tag{5}$$

Similarly, for $y = g(x)$,

$$xg''(x) - 2g'(x) - \frac{4}{x}g(x) = 12 \tag{6}$$

Now substract the last equation from the second last equation, (5)-(6), and combine like terms (namely, the terms with the same order of derivative)

$$(xf''(x) - xg''(x)) + (-2f'(x) + 2g'(x)) + (-\frac{4}{x}f(x) + \frac{4}{x}g(x)) = 0$$

Notice that the RHS is now exactly zero due to the cancellation of 12 and 12. Then, on the LHS, pull out some common factors

$$x(f''(x) - g''(x)) - 2(f'(x) - g'(x)) - \frac{4}{x}(f(x) - g(x)) = 0.$$

Obviously if we set $f(x)-g(x)=y$, *and therefore* $f'(x)-g'(x)=y'$ *and* $f''(x)-g''(x)=y''$ *in the above equation, we arrive at*

$$xy''-2y'-\frac{4}{x}y=0.$$

This is exactly the homogenenous equations (4). *In other words,* $y=f(x)-g(x)$ *solves the homogeneous counterpart.*

(b) (6 points) With the same $f(x)$ and $g(x)$ as in (a), what DE does $y=c_1f(x)+c_2g(x)$ satisfy?

Solution. *Manipulate the equations* f *and* g *satisfies just like how we treat part (a):*

$$c_1\times equation(5)+c_2\times equation(6) \implies \text{(combine like terms)}$$

$$(c_1xf''(x)+c_2xg''(x))+(-c_12f'(x)-c_22g'(x))+(-c_1\frac{4}{x}f(x)-c_2\frac{4}{x}g(x))=12c_1+12c_2.$$

Notice that the RHS is now $12c_2+12c_2$, *which is neither the RHS of* (3) *nor the RHS of* (4). *Then, just like what we did in part (a), the above equation becomes*

$$x(c_1f(x)+c_2g(x))''-2(c_1f(x)+c_2g(x))'-\frac{4}{x}(c_1f(x)+c_2g(x))=12(c_1+c_2).$$

Set $c_1f(x)+c_2g(x)=y(x)$ *in the above equation, we find the answer*

$$xy''-2y'-\frac{4y}{x}=12(c_1+c_2).$$

(c) (6 points) Knowing $y=-2x$ is a particular solution to (3), find a formula for the general solution of (3). You will need the general solution to (4), which can be constructed using information from Prob. 1. **Note** that part (b) is irrelavent here!

Solution. *From Problem 1, we know two* **homogeneous solution** x^4 *and* x^{-1}. *So, the complementary (homogeneous) solution in its general form is*

$$y_h(x)=c_1x^4+c_2x^{-1}.$$

Add the particular **nonhomogeneous** *solution*

$$y_p(x)=-2x$$

to it and we have the answer:

$$y(x)=y_p(x)+y_h(x)=-2x+c_1x^4+c_2x^{-1}$$

is the general solution to the nonhomogeneous DE (3)

$$xy''-2y'-\frac{4y}{x}=12.$$

3. In a fishery, the population is denoted by $P(t)$. Without harvesting, the capacity is 3000 fish and population follows the dynamics $P'(t) = 10^{-4}P(3000 - P)$. Here the unit of time is month. Suppose at $t = 0$, the number of fish is exactly 3000. Then, the owner starts to harvest the fish at a **constant** rate k fish/month so the fish population is governed by

$$P'(t) = 10^{-4}P(3000 - P) - k, \quad P(0) = 3000. \tag{7}$$

In the following questions, we assume the owner uses different formulations for k and examine what will happen to the fish respectively.

(a) (10 points) What is the largest value, denoted by k_{safe}, that one can use for k without endangering the fish? That is to say, if the owner chooses to have $k > k_{safe}$, then equation (7) will have no equilibrium and $P(t)$ will decrease to zero as time t goes by. On the other hand, if the owner chooses to have $k < k_{safe}$, then equation (7) will have 2 equilibria: 1 is stable and the other unstable. And, with $P(0) = 3000$, the population $P(t)$ will approach the stable equilibrium (an expression in terms of k, of course).

Solution. *Set the derivative to be zero in the original DE and we get an algebaic equation for the equilibrium* P_{eq}

$$0 = 10^{-4}P_{eq}(3000 - P_{eq}) - k.$$

This is a quadratic equation in terms of the unknown P_{eq}. *So, in the quadratic formular* $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, *we set* $a = -10^{-4}$, $b = 10^{-4}3000 = 0.3$ *and* $c = -k$,

$$P_{eq} = \frac{-0.3 \pm \sqrt{0.3^2 - 4(-10^{-4})(-k)}}{2(-10^{-4})} \tag{8}$$

In the above equation, the number of **real** *roots depends on the sign of the expression under the* $\sqrt{}$, *i.e., the discriminant*

$$b^2 - 4ac = 0.3^2 - 4(-10^{-4})(-k).$$

So, if the discriminant is positive, then there are two real equilibria, if its negative, there is no real equilibrium and if the discriminant is precisely zero, then there is one real equilibrium. The critical case is

$$discriminant = 0$$

namely

$$0.3^2 - 4(-10^{-4})(-k) = 0$$

The solution to this simple equaion is the value of k_{safe},

$$k_{safe} = 225.$$

(b) (8 points) Draw 3 **slope fields** to illustrate the solution behaviors for the 3 possibilities $k < k_{safe}$, $k = k_{safe}$, $k > k_{safe}$ in (a).

Solution. *c.f. Figure 2.2.8, Figure 2.2.10 and Figure 2.2.11 in the text book for the 3 cases: 2 equilibria, 1 equilibrium and no equilibrium. In the case of 2 P_{eq}, the larger one is stable and the smaller one is unstable. In the case of 1 P_{eq}, it is semi-stable.*

(c) (5 points) Suppose $k = 200$ in (a) which should belong to the safe case $k < k_{safe}$, find the numeric values of the stable and unstable equilibria.

Solution. *Set $k = 200$ in solution formula (8) and obtain*

$$\text{stable:} \quad P \equiv 2000$$
$$\text{unstable:} \quad P \equiv 1000$$

4. Consider a third order linear differential equations

$$\frac{d^3y}{dx^3} - 24\sqrt{3}y = 0$$

(a) (8 points) Verify that

$$y = e^{2\sqrt{3}x}, \quad y = e^{(-\sqrt{3}+3i)x}, \quad y = e^{(-\sqrt{3}-3i)x}$$

are all solutions to this DE. You may use the following identities,

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$
$$i^2 = -1, \quad i^3 = -i, \quad i^4 = 1$$

Solution. *The characteristic equation is*

$$r^3 - 24\sqrt{3} = 0.$$

Set $r = 2\sqrt{3}$ in the above equation and we find it is a root. Therefore, $y = e^{2\sqrt{3}x}$ is a solution. Set $r = -\sqrt{3} + 3i$ in the above characteristic equation and use the cubic identities, e.g. the first identity with $a = -\sqrt{3}$ and $b = 3i$

$$(-\sqrt{3}+3i)^3 = (-\sqrt{3})^3 + 3(-\sqrt{3})^2(3i) + 3(-\sqrt{3})(3i)^2 + (3i)^3 = 24\sqrt{3}.$$

So, $r = -\sqrt{3}+3i$ is also a solution to the characteristic equation $r^3 - 24\sqrt{3}$ and therefore $y = e^{(-\sqrt{3}+3i)x}$ is a solution to the original DE. Similarly for the other complex root (indeed, we simply need to use the fact that complex roots always appear in conjugate pair for real coef polynomials.)

(b) (6 points) Use (a) to show that

$$y = e^{-\sqrt{3}x}\sin(3x), \quad y = e^{-\sqrt{3}x}\cos(3x)$$

are also solutions to this DE. The simplest way is to use the Euler's identity

$$e^{r+\theta i} = e^r e^{i\theta} = e^r(\cos(\theta) + i\sin(\theta))$$

to transform the two complex solutions in (a) from exponential form into trigonometric form. Then, it is easy to speculate that the solutions in part (b) are superpositions of solutions in (a).

Solution. *Apply the Euler's identity to rewrite the second solution in part (a) as*

$$y = e^{(-\sqrt{3}+3i)x} = e^{(-\sqrt{3})x}(\cos(3x) + i\sin(3x))$$

and the third solution in part (a)

$$y = e^{(-\sqrt{3}-3i)x} = e^{(-\sqrt{3})x}(\cos(3x) - i\sin(3x)).$$

(Note that the sign-flipping of the imaginary parts in the above calculation is alwyas true for complex solutions appearing in conjugate pair.) Then, we add the above equations and divide it by 2

$$\frac{e^{(-\sqrt{3}+3i)x} + e^{(-\sqrt{3}-3i)x}}{2} = e^{-\sqrt{3}x}\cos(3x).$$

So, $e^{-\sqrt{3}x}\cos(3x)$ *is a linear combination of the exponential solutions* $e^{(-\sqrt{3}+3i)x}$ *and* $e^{(-\sqrt{3}-3i)x}$ *and it is therefore also a solution (the DE in this problem is obviously linear AND homogeneous).*

To verify $e^{-\sqrt{3}x}\sin(3x)$ *is yet another solution, we use*

$$\frac{e^{(-\sqrt{3}+3i)x} - e^{(-\sqrt{3}-3i)x}}{2i} = e^{-\sqrt{3}x}\sin(3x).$$

(c) (8 points) Find the particular solution satisfying

$$y(0) = -3, \quad y'(0) = -3, \quad y(\frac{\pi}{3}) = 2e^{-\sqrt{3}\pi/3} - e^{2\sqrt{3}\pi/3}.$$

You should start with picking **three** linearly independent solutions out of the five candidates in (a) and (b) and then superimpose them to generate the general solution.

Solution. *Since all give data are real, it is probably easier to use the two trigonometric solutions in part (b) together with the only real solution in part (a) to assemble the general solution*

$$y(x) = c_1 e^{-\sqrt{3}x}\cos(3x) + c_2 e^{-\sqrt{3}x}\sin(3x) + c_3 e^{2\sqrt{3}x}.$$

Plug in the given data and solve for the coef c_1, c_2, c_3.

Answer:

$$y = -e^{2\sqrt{3}x} - e^{-\sqrt{3}x}\sin(3x) - 2e^{-\sqrt{3}x}\cos(3x).$$