MAT274 Homework 3

Due: Friday 9/17

Readings: Chapter 3 of *Edwards & Penney*

Points: 12'+18'+23'+22'=75'

1. (12 points) Knowing that $y = x^4$ and $y = \frac{1}{x}$ are both solutions to

$$xy'' - 2y' - \frac{4y}{x} = 0,$$

prove that they are linearly independent on $x \in (0, \infty)$ and $x \in (-\infty, 0)$. Then, prove that any initial conditions of the form

$$y(x_0) = y_1, \quad y'(x_0) = y_2$$

with $x_0 \neq 0$ will guarantee a particular solution for either $x \in (0, \infty)$ or $x \in (-\infty, 0)$.

2. Consider a **nonhomogenenous** linear equation

$$xy'' - 2y' - \frac{4y}{x} = 12 \tag{1}$$

(a) (6 points) If $y = f(x)$ and $y = g(x)$ are two solutions to the above DE, prove that $y = f(x) - g(x)$ satisfies

$$xy'' - 2y' - \frac{4y}{x} = 0 \tag{2}$$

which is the **homogeneous** counterpart of the nonhomogeneous DE (1).

(b) (6 points) With the same $f(x)$ and $g(x)$ as in (a), what DE does $y = c_1 f(x) + c_2 g(x)$ satisfy?

(c) (6 points) Knowing $y = -2x$ is a particular solution to (1), find a formula for the general solution of (1). You will need the general solution to (2), which can be constructed using information from Prob. 1. **Note** that part (b) is irrelavent here!

3. In a fishery, the population is denoted by $P(t)$. Without harvesting, the capacity is 3000 fish and population follows the dynamics $P'(t) = 10^{-4}P(3000 - P)$. Here the unit of time is month. Suppose at $t = 0$, the number of fish is exactly 3000. Then, the owner starts to harvest the fish at a **constant** rate k fish/month so the fish population is governed by

$$P'(t) = 10^{-4}P(3000 - P) - k, \quad P(0) = 3000. \tag{3}$$

In the following questions, we assume the owner uses different formulations for k and examine what will happen to the fish respectively.

(a) (10 points) What is the largest value, denoted by k_{safe}, that one can use for k without endangering the fish? That is to say, if the owner chooses to have $k > k_{safe}$, then equation (3) will have no equilibrium and $P(t)$ will decrease to zero as time t goes by. On the other hand, if the owner chooses to have $k < k_{safe}$, then equation (3) will have 2 equilibria: 1 is stable and the other unstable. And, with $P(0) = 3000$, the population $P(t)$ will approach the stable equilibrium (an expression in terms of k, of course).

(b) (8 points) Draw 3 **slope fields** to illustrate the solution behaviors for the 3 possibilities $k < k_{safe}$, $k = k_{safe}$, $k > k_{safe}$ in (a).

(c) (5 points) Suppose $k = 200$ in (a) which should belong to the safe case $k < k_{safe}$, find the numeric values of the stable and unstable equilibria.

4. Consider a third order linear differential equations

$$\frac{d^3y}{dx^3} - 24\sqrt{3}y = 0$$

(a) (8 points) Verify that

$$y = e^{2\sqrt{3}x}, \quad y = e^{(-\sqrt{3}+3i)x}, \quad y = e^{(-\sqrt{3}-3i)x}$$

are all solutions to this DE. You may use the following identities,

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$i^2 = -1, \quad i^3 = -i, \quad i^4 = 1$$

(b) (6 points) Use (a) to show that

$$y = e^{-\sqrt{3}x}\sin(3x), \quad y = e^{-\sqrt{3}x}\cos(3x)$$

are also solutions to this DE. The simplest way is to use the Euler's identity

$$e^{r+\theta i} = e^r e^{i\theta} = e^r(\cos(\theta) + i\sin(\theta))$$

to transform the two complex solutions in (a) from exponential form into trigonometric form. Then, it is easy to speculate that the solutions in part (b) are superpositions of solutions in (a).

(c) (8 points) Find the particular solution satisfying

$$y(0) = -3, \quad y'(0) = -3, \quad y(\frac{\pi}{3}) = 2e^{-\sqrt{3}\pi/3} - e^{2\sqrt{3}\pi/3}.$$

You should start with picking **three** linearly independent solutions out of the five candidates in (a) and (b) and then superimpose them to generate the general solution.