MAT274   HW2: SOLUTIONS

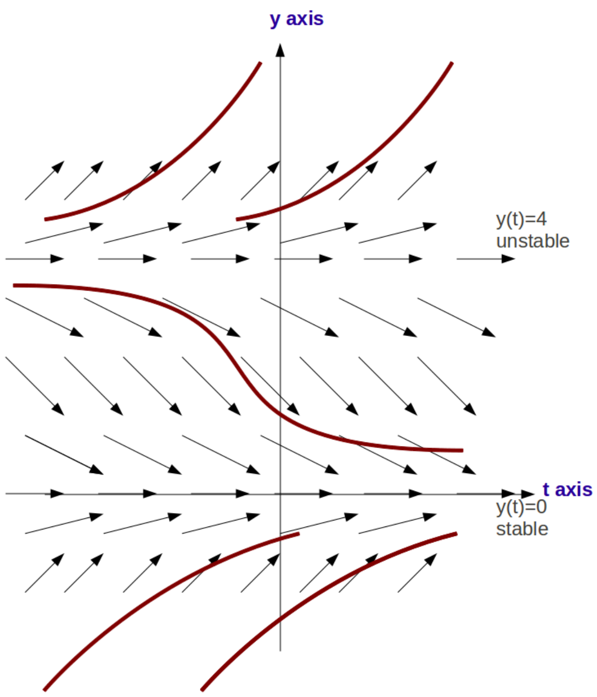
DUE IN CLASS, SEPTEMBER 8, 2010.
READINGS: §1.6, 2.1 – 2.3 OF EDWARDS & PENNEY

1. Consider the differential equation

$$\frac{dy}{dt} = y(y-4).$$

i) Draw its slope field.

Solution.



ii) Describe the behavior of $y(t)$ as $t \to \infty$. Its asymptotic behavior may depend on the initial condition and you should specify all types (increase, decrease, constant, etc).

Solution. *When the initial condition* $y(t_0) = y_0 < 0$*, the solution increases and* *approaches 0*

$$\lim_{t\to\infty} y(t) = 0.$$

When the initial condition $y(t_0) = y_0 = 0$*, the solution stays at constant 0*

$$y(t) \equiv 0.$$

When the initial condition $y(t_0) = y_0 \in (0, 4)$*, the solution decreases and approaches 0*

$$\lim_{t\to\infty} y(t) = 0.$$

When the initial condition $y(t_0) = y_0 = 4$*, the solution stays at constant 4*

$$y(t) \equiv 4$$

When the initial condition $y(t_0) = y_0 > 4$*, the solution increases and approaches infinity*

$$\lim_{t\to\infty} y(t) = \infty$$

iii) Find the general solution for $y(t)$. Is your result consistent with ii)?

Solution. *Separation of variables*

$$\frac{dy}{y(y-4)} = dt.$$

Integrate both sides

$$\int \frac{1}{y(y-4)} dy = \int dt. \tag{1}$$

On the LHS, use partial fractions

$$\frac{1}{y(y-4)} = \frac{A}{y} + \frac{B}{y-4}$$

where A, B *are undetermined coefficients. Multiply through with* $y(y-4)$,

$$1 = A(y-4) + By$$

that is

$$1 = (A+B)y - 4A.$$

Comparing the coef, we have

$$A + B = 0 \quad \textit{and} \quad -4A = 1$$

from which we solve for A, B,

$$A = -1/4, \qquad B = 1/4.$$

Therefore, we establish the partial fractions for the integrand on the LHS

$$\int \frac{1}{y(y-4)}dy = \int -\frac{1}{4y}dy + \int \frac{1}{4(y-4)}dy$$

and upon integration

$$\int \frac{1}{y(y-4)}dy = -\frac{1}{4}\ln|y| + \frac{1}{4}\ln|y-4| + C = \frac{1}{4}\ln\left|\frac{y-4}{y}\right| + C$$

The RHS of (1) *in the previous page is* $t + C$. *So, the implicit solution is*

$$\frac{1}{4}\ln\left|\frac{y-4}{y}\right| = t + C$$

Solve for y *to get an explicit form*

$$\left|\frac{y-4}{y}\right| = e^{4t+4C}$$

i.e.

$$\frac{y-4}{y} = \pm e^{4t+4C} = C_1 e^{4t}.$$

Here, we let the new constant $C_1 = \pm e^{4C}$. *So, finally,*

$$y(t) = \frac{4}{1 - C_1 e^{4t}}.$$

The following analysis is not essential to the syllabus of this course but can help understand the concepts.

The behavior of a particular solution depends on several factors:

1. If $C_1 > 0$, *then, solution is a increasing funcion of* t*; if* $C_1 < 0$, *the solution is a decreasing function of* t*; if* $C_1 = 0$, *then solution is constant.*

2. In the case when $C_1 > 0$ *and the initial condition* $y(t_0) > 4$, *we have*

$$\frac{4}{1 - C_1 e^{4t_0}} > 4 \implies 1 - C_1 e^{4t_0} > 0.$$

And since $\frac{4}{1-C_1 e^{4t}}$ *is a decreasing function of* t, *it will get closer and closer to 0 (but still remains positive). In such case, the limit of* $y(t)$ *becomes infinity. For all other combinations of* C_1 *and* y_0, *the solution either approaches zero or remains constant.*

2. It is the pollen season. Consider the air in an apartment with volume V. Let the internal concentration of pollen be $x(t)$ and the ambient (external) concentration $a(t)$. Assume, inside the apartment, pollen is evenly distributed so that the total amount of pollen is given by

$$P(t) = V \cdot x(t).$$

Let air flow into and out of the apartment at a rate of $r(t)$.

i) How much air flows in and out during a time period dt? How much pollen is in and out, respectively?

Solution. *"Air-in" and "air-out" are both $r(t)dt$. "Pollen-in" is the ambient concentration times "air-in":*

$$a(t)r(t)dt.$$

"Pollen-out" is the internal concentration times "air-out":

$$x(t)r(t)dt$$

ii) Find the corresponding change in the total amount of pollen $dP(t)$.

Solution. *Change in the total amount of pollen is the difference between "pollen-in" and "pollen-out"*

$$dP(t) = a(t)r(t)dt - x(t)r(t)dt$$

iii) Write down a differential equation for the internal concentration $x(t)$.

Solution. *Rate of change in the* **total** *amount of pollen is $dP(t)/dt$ and by part ii), we have the equation*

$$\frac{dP(t)}{dt} = a(t)r(t) - x(t)r(t). \tag{2}$$

Now, $P(t)$ is related to $x(t)$ as

$$P(t) = Vx(t)$$

which is given in the problem. So, $dP/dt = V dx/dt$ and plug it into equation (2) and arrive at

$$V\frac{dx(t)}{dt} = a(t)r(t) - x(t)r(t). \tag{3}$$

This is a "closed" equation that has only one unknown $x(t)$ (whereas equation (2) has two: $x(t)$ and $P(t)$.)

iv) Suppose $a(t), r(t)$ are known. Propose an applicable method to solve for $x(t)$. DO NOT actually solve it.

Solution. *Equation (3) is linear in $x(t)$ and $x'(t)$, so we can use integrating factor. That is, rewrite the DE into a "standard form"*

$$x'(t) + \frac{r(t)}{V}x(t) = \frac{a(t)r(t)}{V}$$

and introduce

$$\mu(t) = e^{\int \frac{r(t)}{V} dt}$$

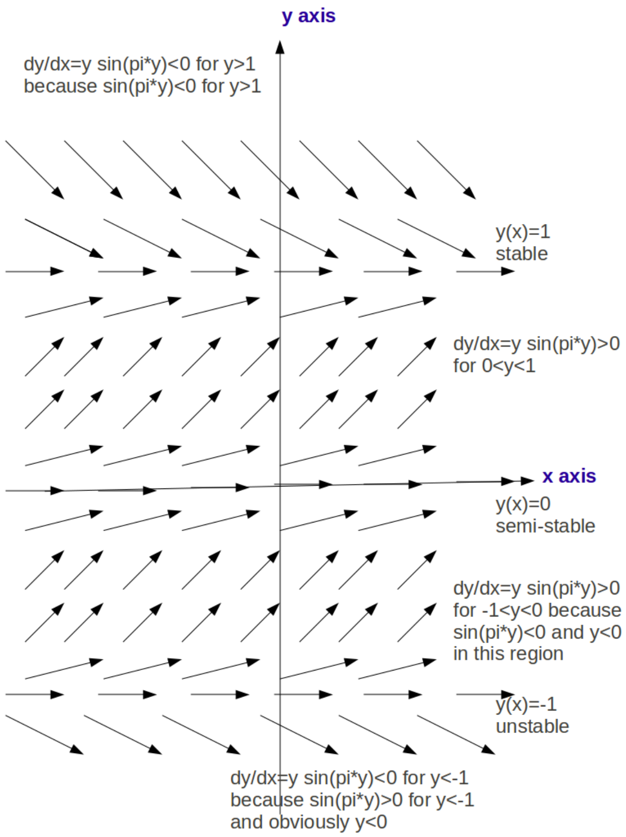
and multiply the DE with μ. The LHS should become a perfect derivative and can be integrated easily.

3. Find **all** equilibrium solutions for

$$\frac{dy}{dx} = y\sin(\pi y), \qquad y \in (-3/2, 3/2)$$

and classify them as asymptotically stable, unstable or semi-stable. Use slope field to support your argument.

Solution. *Equilibrium* $y(x) \equiv 0$ *is semistable;* $y(x) \equiv 1$ *is stable; and* $y(x) \equiv -1$ *is unstable.*



4. Using the substitution $v = y/x$ to transform the following DE into another DE in terms of unknown function $v(x)$. Then, solve for $v(x)$ in this new DE and finally find the general solution $y(x)$ of the original equation.

$$(x - 3y)y' = (y + x).$$

Solution. *First, try to rearrange the DE so that every term is either in terms of* dy/dx *or in terms of* y/x*. To achieve this, divide through with* $x - 3y$

$$\frac{dy}{dx} = \frac{y + x}{x - 3y}.$$

Then, divide the lower and upper parts of the RHS with x,

$$\frac{dy}{dx} = \frac{\frac{y}{x}+1}{1-3\frac{y}{x}}. \tag{4}$$

If $v = y/x$, then $y = xv$. Regard $y(x)$ and $v(x)$ as functions of x and apply the product rule

$$\frac{dy}{dx} = \frac{d}{dx}(xv) = \frac{dx}{dx}v + x\frac{dv}{dx} = v + x\frac{dv}{dx}.$$

Substitute this for y' in equation (4) and replace $\frac{y}{x}$ with v as well,

$$v + x\frac{dv}{dx} = \frac{v+1}{1-3v}.$$

Move the first term on the LHS to RHS

$$x\frac{dv}{dx} = \frac{v+1}{1-3v} - v$$

and then apply separation of variables

$$\frac{dv}{\frac{v+1}{1-3v} - v} = \frac{dx}{x}$$

that is

$$\frac{1-3v}{1+3v^2}dv = \frac{1}{x}dx. \tag{5}$$

Integrate the LHS

$$\begin{aligned}
\int \frac{1-3v}{1+3v^2}dv &= \int \frac{1}{1+3v^2}dv - \int \frac{3v}{1+3v^2}dv \\
&= \int \frac{1}{1+w^2}\frac{1}{\sqrt{3}}dw - \int \frac{1}{u}\frac{1}{2}du \qquad \textit{let } w=\sqrt{3}v \textit{ in the first term} \\
&\qquad\qquad \textit{and let } u = 1+3v^2 \textit{ in the second term} \\
&= \frac{1}{\sqrt{3}}\tan^{-1} w - \frac{1}{2}\ln|u| + C \qquad \textit{the first term coming from table of integrals} \\
&= \frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}v) - \frac{1}{2}\ln|1+3v^2| + C.
\end{aligned}$$

Obviously, the RHS of equation (5) integrates to $\ln|x|$. So, we have

$$\frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}v) - \frac{1}{2}\ln|1+3v^2| = \ln|x| + C.$$

Do not forget the last step: replace v with y/x in the above expression

$$\frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}\frac{y}{x}) - \frac{1}{2}\ln|1+\frac{3y^2}{x^2}| = \ln|x| + C.$$

5. Let an object (with mass m) start a vertical motion near the surface of the Earth with initial velocity v_0. Let the resistance be proportional to v^p, that is $F_R = -kv|v|^{p-1}$

   i) Briefly explain why we use an absolute value in the formula for F_R.

Solution. *Because the drag force always opposites the velocity, F_R and v are of opposite signs. Thus, the minus sign together with the absolute sign gurantees $F_R = -kv|v|^{p-1}$ is always of opposite sign to v.*

   ii) Write down an DE for the velocity v.

Solution. *The gravitational force is in the negative, vertical direction and is constant,*

$$\frac{dv}{dt} = -kv|v|^{p-1} - mg.$$

   iii) Without solving the DE, use a slope field to study the asymptotic behavior of $v(t)$ as $t \to \infty$. In particular, find the terminal speed. Does it matter if the initial velocity $v_0 > 0$ or $v_0 < 0$?

Solution. *We did this in class. The ONLY equilibrium solution is*

$$v_{eq} = -\left(\frac{mg}{k}\right)^{\frac{1}{p}}$$

and is a stable equilibrium. So, no matter what the initial condition is, the limit of $v(t)$ as $t \to \infty$ is always the same

$$\lim_{t\to\infty} v(t) = -\left(\frac{mg}{k}\right)^{\frac{1}{p}}.$$