MAT274 Homework 1

Due: Wed 9/1

Readings: §1.1, 1.3 – 1.5 of *Edwards & Penny*

Points: 10 points each for P1–P5; 20 points each for P6–P7.

1. The equation
$$x\frac{dy}{dx} + 3y = 2x^5$$
is a <u>first</u> order differential equation that is <u>linear</u> (linear/nonlinear). Verify that $y(x) = \frac{1}{4}x^5 + Cx^{-3}$ with constant C is a general solution. Then, determine the value of C using initial condition $y(2) = 1$.
Solution. For $y(x) = \frac{1}{4}x^5 + Cx^{-3}$,
$$y'(x) = \frac{5}{4}x^4 - 3Cx^{-4}.$$
Plug it into the LHS of the original DE
$$xy' + 3y = x(\frac{5}{4}x^4 - 3Cx^{-4}) + 3(\frac{1}{4}x^5 + Cx^{-3})$$
which is simplified to
$$xy' + 3y = 2x^5.$$
For the particular solution, set $x = 2, y = 1$ in the general solution $y(x) = \frac{1}{4}x^5 + Cx^{-3}$
$$1 = \frac{1}{4} \cdot 2^5 + C \cdot 2^{-3}$$
and solve for C
$$C = -56.$$

2. Use the *Matlab* command `dfield8` to help sketch the slop field and *several* solutions of the previous problem. Then, indicate the curve of the particular solution that satifies the given initial condition $y(1) = 3$.

3. Suppose a population P is a function of time t. If the birth rate is 2 times the square of P and the death rate is a constant 3, what is the differential equation that models the population dynamics? Is the equation linear or nonlinear? How many initial conditions is needed to determine a specific solution?
Solution. $P'(t) = 2P^2 - 3$. It's nonlinear. One initial condition is needed.

4. **Without** help of computers or calculators, sketch the slope field and *several* typical solutions of the differential obtained from previous problem. Indicate the curve that satisfes $y(0) = 1$ and the curve that satisfes $y(0) = 2$. Is the population increasing or decreasing according to these two curves? (The skill of sketching slope fields **by hand** is required in this class).
Solution. The population is increasing with initial condition $y(0) = 2$ and increasing with $y(0) = 1$.

5. Give an example of a 2nd order, linear, homogeneous differential equation with constant coefficients. Specify 2 initial conditions, too.
Solution.
$$3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - 4y = 0,$$
$$y(0) = -1, \quad y'(0) = 2.$$

6. Find general solutions (implicit if necessary, explicit if convenient) of the following equations

   (a) $y' = y^3/x$;
   Solution. Separation of variables.
   $$\frac{dy}{dx} = \frac{y^3}{x}$$
   Multiply with dx and divide with y^3
   $$\frac{dy}{y^3} = \frac{dx}{x}$$
   Integrate both sides
   $$\int y^{-3} dy = \int x^{-1} dx$$
   that is
   $$-\frac{1}{2}y^{-2} = \ln|x| + C$$

and solve for y

$$y(x) = \pm(-2\ln|x| - 2C)^{-1/2}$$

(b) $\frac{dy}{dx} = e^{2y-3}\cos x$.

Solution. Separation of variables.

$$\frac{dy}{dx} = e^{2y-3}\cos x$$

Multiply with dx and divide with e^{2y-3}

$$\frac{dy}{e^{2y-3}} = \cos x dx$$

Integrate both sides

$$\int \frac{dy}{e^{2y-3}} = \int \cos x dx$$

The LHS integral is

$$\int \frac{dy}{e^{2y-3}} = \int e^{3-2y} dy = -\frac{1}{2}e^{3-2y}$$

and the RHS integral is $\sin x + C$. So,

$$-\frac{1}{2}e^{3-2y} = \sin x + C \implies e^{3-2y} = -2\sin x - 2C$$

Take the ln on both sides

$$3 - 2y = \ln(-2\sin x - 2C)$$

that is

$$y = 1.5 - \frac{1}{2}\ln(-2\sin x - 2C)$$

7. Find particular solutions of

(a) $xy' - 2y = x^3, \qquad y(2) = 8;$

Solution. Rewrite the equation into a standard form so that the coef of y' is one and all terms involving y are moved to the LHS

$$y' - \frac{2}{x}y = x^2 \tag{1}$$

Integrating factor

$$\mu(x) = e^{\int -\frac{2}{x}dx} = e^{-2\ln|x|}$$

Since $-2\ln|x| = \ln(|x|^{-2}) = \ln(x^{-2})$, we have

$$\mu(x) = e^{\ln(x^{-2})} = x^{-2}.$$

Multiply equation (1), i.e. the standard form, with $\mu(x) = x^{-2}$

$$x^{-2}y' - 2x^{-3}y = 1.$$

As the consequence of using integrating factor, the LHS of the above DE has to equal a perfect derivative,

$$\frac{d}{dx}(x^{-2}y) = 1$$

<u>**Note**</u> that the function being differentiated is always the integrating factor $\mu(x)$ times the unknown $y(x)$.
Finally, the last DE above can be integrated easily

$$x^{-2}y = x + C.$$

Plug in the initial condition $x = 2, y = 8$,

$$2^{-2} \cdot 8 = 2 + C \implies C = 0$$

Thus, the particular solution is $x^{-2}y = x$, namely

$$y = x^3.$$

(b) $y' = e^{(x^2)} + y + 2xy, \qquad y(0) = 1.$
Solution. Rewrite the equation into a standard form so that the coef of y' is one and all terms involving y are moved to the LHS

$$y' - (1 + 2x)y = e^{(x^2)} \tag{2}$$

Integrating factor

$$\mu(x) = e^{\int -(1+2x)dx} = e^{-x-x^2}$$

Multiply equation (2), i.e. the standard form, with $\mu(x) = e^{-x-x^2}$

$$e^{-x-x^2}y' - e^{-x-x^2}(1 + 2x)y = e^{-x}.$$

As the consequence of using integrating factor, the LHS of the above DE has to equal a perfect derivative,

$$\frac{d}{dx}(e^{-x-x^2}y) = e^{-x}$$

Note that the function being differentiated is always the integrating factor $\mu(x)$ times the unknown $y(x)$.
Finally, the last DE above can be integrated easily

$$e^{-x-x^2}y = -e^{-x} + C.$$

Plug in the initial condition $x = 0, y = 1$,

$$e^0 \cdot 1 = -e^0 + C \implies C = 2$$

Thus, the particular solution is $e^{-x-x^2}y = -e^{-x} + 2$, namely

$$y = -e^{(x^2)} + 2e^{(x+x^2)}.$$