MAT274 HW10

DUE DECEMBER 7 (OR SUBMIT IN MONDAY CLASS).
READINGS: §7.1—7.3 OF EDWARDS & PENNEY

1. Let $f(t) = t^n$ where n is a positive integer. Apply **definition** and perform a detailed calculation to find its Laplace transform $F(s)$.
   Solution. By definition $L[t^n] = \int_0^\infty e^{-st}t^n\,dt$. Perform integration by parts with $u = t^n$, $dv = e^{-st}$ and $du = nt^{n-1}$, $v = -\frac{1}{s}e^{-st}$

$$L[t^n] = -\frac{1}{s}e^{-st}t^n\Big|_{t=0}^{t=\infty} - \int_0^\infty -\frac{1}{s}e^{-st}nt^{n-1}\,dt$$

$$= \frac{n}{s}\int_0^\infty e^{-st}t^{n-1}\,dt$$

since $\lim_{t\to\infty} e^{-st}t^n = 0$ for any $s > 0$. Now, by definition of Laplace transform again, the RHS above amounts to $\frac{n}{s}L[t^{n-1}]$ and thus

$$L[t^n] = \frac{n}{s}L[t^{n-1}]$$

This relation can be applied recursively to

$$L[t^{n-1}] = \frac{n-1}{s}L[t^{n-2}]$$

$$L[t^{n-2}] = \frac{n-2}{s}L[t^{n-3}]$$

etc. In other words, each time we reduce the power of t by 1, we introduce a factor of $\frac{k}{s}$ where k is the power of t. So

$$L[t^n] = \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}......\frac{1}{s}L[t^0] = \frac{n!}{s^n}L[1]$$

Finally, $L[1] = 1/s$ so

$$L[t^n] = \frac{n!}{s+1}$$

2. Use the Table of Laplace Transforms and the property

$$\mathcal{L}\{(-t)^n f(t)\} = \frac{d^n}{ds^n}\mathcal{L}\{f(t)\}$$

to find

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+9)^2}\right\}$$

Hint: think about $\frac{d}{ds}\frac{1}{s^2+9}$.

Solution. Start with using the following item from the table $L[\sin kt] = \frac{k}{s^2+k^2}$ and set $k = 3$

$$L[\sin 3t] = \frac{3}{s^2+9}.$$

Divide both sides with 3 and use the linearity of L

$$L[\frac{1}{3}\sin 3t] = \frac{1}{s^2+9}$$

By the property given in the problem,

$$L[-t \cdot \frac{1}{3}\sin 3t] = \frac{d}{ds}\frac{1}{s^2+9} = -\frac{2s}{(s^2+9)^2}$$

So, again, by linearity of L,

$$L[\frac{1}{2} \cdot t \cdot \frac{1}{3}\sin 3t] = \frac{s}{(s^2+9)^2}$$

namely

$$L^{-1}\left[\frac{s}{(s^2+9)^2}\right] = \frac{1}{6}t\sin 3t.$$

3. Consider an undamped spring-mass system with spring constant 64. At time $t = 0$, the mass is released at displacement +2 with initial velocity -4. Use **Laplace transform** to find the particular solution. You may use the Table of Laplace Transforms directly.

Solution. For simplicity, let the mass $m = 1$. Then, the DE is

$$x''(t) + 64x(t) = 0, \quad x(0) = 2, \quad x'(0) = -4$$

Let $X(s) = L[x(t)]$. Then, the Laplace Transform of its second derivative satisfies

$$L[x''(t)] = s^2X(s) - sx(0) - x'(0)$$

Then, under Laplace transform, the DE $x'' + 64x = 0$ becomes

$$s^2X(s) - sx(0) - x'(0) + 64X(s) = 0.$$

Apply the initial conditions $x(0) = 2$, $x'(0) = -4$ and arrive at

$$s^2X(s) - 2s + 4 + 64X(s) = 0.$$

From here, we easily solve for $X(s)$

$$X(s) = \frac{2s-4}{s^2+64}.$$

Now, use the Table of Laplace Transform, we know $L^{-1}\left[\frac{s}{s^2+64}\right] = \cos 8t$ and $L^{-1}\left[\frac{8}{s^2+64}\right] = \sin 8t$. Also, by the linearity of L^{-1},

$$L^{-1}[X(s)] = L^{-1}\left[\frac{2s-4}{s^2+64}\right] = 2L^{-1}\left[\frac{s}{s^2+64}\right] - \frac{1}{2}L^{-1}\left[\frac{8}{s^2+64}\right]$$

Therefore,

$$x(t) = L^{-1}[X(s)] = 2\cos 8t - 0.5\sin 8t.$$