# SINGULAR LIMITS AND CONVERGENCE RATES OF COMPRESSIBLE EULER AND ROTATING SHALLOW WATER EQUATIONS[*]

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**Abstract.** With solid-wall boundary condition and ill-prepared initial data, we prove the singular limits and convergence rates of compressible Euler and rotating shallow water equations towards their incompressible counterparts. A major issue is that fast acoustic waves contribute to the slow vortical dynamics at order one and do not damp in any strong sense. Upon *averaging in time*, however, such a contribution vanishes at the order of the singular parameters (i.e., Mach/Froude/Rossby numbers). In particular, convergence rates of the compressible dynamics, when projected onto the slow manifold, are estimated explicitly in terms of the singular parameters and Sobolev norms of the initial data. The structural condition of a vorticity equation plays a key role in such an estimation as well as in proving singular-parameter-independent life spans of classical solutions.

**Key words.** compressible Euler equations, rotating shallow water equations, singular limits, convergence rates, initial-boundary value problem, ill-prepared initial data, time averages

**AMS subject classifications.** Primary, 35Q31; Secondary, 35L50, 76N15

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**1. Introduction.** Hyperbolic partial differential equations (PDEs) of multiscale nature have seen rapidly growing applications in recent years. We conduct a theoretical investigation in this paper concerning two prototypical examples: the compressible Euler equations with low Mach number and the rotating shallow water (RSW) equations with low Froude and Rossby numbers. All three numbers depend on physical models. Here, we study the scaling regime where they are small and thus give rise to multiscale dynamics. The settings of these two systems are given below.

The spatial domain $\Omega \subset \mathbf{R}^{\mathbf{D}}$ ($\mathbf{D} = 2$ or $3$) is bounded and connected, and its boundary $\partial\Omega$ is smooth and connected. Prescribe the solid-wall boundary condition on the velocity field $u$,

$$u \cdot \mathsf{n}\big|_{\partial\Omega} = 0.$$

Here and below, $\mathsf{n} = \mathsf{n}(x)$ denotes the outward normal at $x \in \partial\Omega$.

Both Euler and RSW equations can be described in terms of *total* density $\hat{\rho}$ and velocity $u$,

$$\begin{cases} \partial_t \hat{\rho} + \nabla \cdot (\hat{\rho} u) = 0, \\ \hat{\rho}(\partial_t u + u \cdot \nabla u) + \dfrac{\nabla p(\hat{\rho})}{\varepsilon^2} = \dfrac{1}{\varepsilon} F, \end{cases}$$

where small parameter $\varepsilon \ll 1$ brings in fast oscillations. The equation of state is

$$\text{pressure} = p(\hat{\rho}) \quad \text{with} \quad p(\cdot) \in C^\infty,\ p(1) = 0,\ p'(1) = 1.$$

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The differences between these two systems are the following. For Euler equations, we consider two or three dimensions and external force $F = 0$. For RSW equations, only dimension $\mathbf{D} = 2$ (on the horizontal plane) is considered and external forcing is due to the Coriolis effect[1] $F = -u^\perp$. Also, $\hat{\rho}$ is understood as the total thickness of water/air in the vertical direction.

Without loss of generality, impose

$$\frac{1}{|\Omega|} \int_\Omega \hat{\rho} \, dx = 1$$

since $\int_\Omega \hat{\rho} \, dx$ is conserved by both systems.

Having $\hat{\rho} \approx 1$ as the total density, define the density perturbation

$$(1.1) \qquad \rho := \frac{\hat{\rho} - 1}{\varepsilon},$$

and rewrite the above system for the unknown pair $(\rho, u)$,

$$(1.2) \qquad \begin{cases} \partial_t \rho + \nabla \cdot (\rho u) + \dfrac{\nabla \cdot u}{\varepsilon} = 0, \\[2mm] \partial_t u + u \cdot \nabla u + \dfrac{p'(1 + \varepsilon \rho)}{1 + \varepsilon \rho} \dfrac{\nabla \rho}{\varepsilon} = \dfrac{1}{\varepsilon} F, \end{cases}$$

$$(1.3) \qquad u \cdot \mathsf{n}\big|_{\partial\Omega} = 0, \qquad \int_\Omega \rho \, dx = 0.$$

THEOREM 1.1 (Euler equations). *Consider the initial-boundary value problem of the* $\mathbf{D}$*-dimensional compressible Euler equations* (1.2) *with* $F = 0$ *and constraints* (1.3) *subject to initial data* $(\rho_0, u_0) \in H^m(\Omega)$ *with* $m > \frac{\mathbf{D}}{2} + 4$*. Assume* $(\rho_0, u_0)$ *is compatible with the boundary condition "*$\partial_t^k u_0$*" $\cdot \mathsf{n}\big|_{\partial\Omega} = 0$ for $k < m$—consult Remark 1.2.*

*Then, upon Helmholtz decomposition of the solution into incompressible and potential parts (detailed definition given in section 2),* $(\rho, u) = (\rho^P, u^P) + (\rho^Q, u^Q)$*, there exist general constants* $C, T$ *dependent only on* $m, \Omega, p(\cdot)$*, and* $\|(\rho_0, u_0)\|_{H^m}$ *so that*

$$(1.4) \qquad \rho^P \equiv 0, \qquad \max_{0 \le t \le T} \|u^P - \overline{u}\|_{H^{m-3}(\Omega)} \le C\varepsilon,$$

*where* $\overline{u}$ *solves the incompressible Euler equations*

$$(1.5) \qquad \begin{cases} \partial_t \overline{u} + \overline{u} \cdot \nabla \overline{u} + \nabla q = 0, \\ \nabla \cdot \overline{u} = 0, \end{cases}$$
$$\overline{u} \cdot \mathsf{n}\big|_{\partial\Omega} = 0, \qquad \overline{u}_0 = u_0^P.$$

*Moreover, the fast component of the solution vanishes at order* $O(\varepsilon)$ *in the sense of time averaging:*

$$(1.6) \qquad \left\| \frac{1}{t} \int_0^t (\rho^Q, u^Q) d\tau \right\|_{H^m(\Omega)} \le C\varepsilon \left( 1 + \frac{1}{t} \right) \qquad \text{for } t \in (0, T].$$

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[1] In this article, $(\,\dot{}\,)^\perp$ denotes a left multiplication by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, e.g., $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}^\perp$ and, in two dimensions, $\nabla^\perp$.

*Proof* (sketch of proof). The key ingredients are the following. For (1.6), see Theorem 2.5, which is based on Lemmas 2.4 and 2.6. For (1.4) and (1.5), see Theorem 3.1, which is based on Lemmas 3.2 and 3.3. ☐

REMARK 1.2. *In the compatibility condition, "$\partial_t^k u_0$" is obtained from expressing $\partial_t^k u$ in terms of spatial derivatives via (1.2) and then substituting $(\rho_0, u_0)$ into the expression. This condition is crucial in obtaining $H^m(\Omega)$ regularity in short time, although well-posedness is not the issue of focus in our result. We also note in passing that it is proved in [19] that this condition is indeed necessary for the well-posedness of certain linear hyperbolic PDEs.*

We also remark that, in this theorem and the one below, the $(\rho^P, u^P)$ component is slow, as its time derivative can be bounded independent of $\varepsilon$. On the other hand, the fast component $(\rho^Q, u^Q)$ has a time derivative of order $O(\varepsilon^{-1})$.

THEOREM 1.3 (RSW equations). *Consider the RSW equations (1.2) with $F = -u^\perp$ and constraints (1.3) in domain $\Omega \subset \mathbf{R}^2$. Under the same assumptions as in Theorem 1.1, there exists a projection of the solution onto slow and fast manifolds (detailed definition given in section 5), $(\rho, u) = (\rho^P, u^P) + (\rho^Q, u^Q)$, so that for general constants $C, T$ dependent only on $m, \Omega, p(\cdot)$, and $\|(\rho_0, u_0)\|_{H^m}$,*

$$(1.7) \qquad (\rho^Q, u^Q) \text{ vanishes at order } O(\varepsilon) \text{ in the same sense as (1.6)}$$

*and*

$$(1.8) \qquad \max_{0 \le t \le T} \|(\rho^P, u^P) - (\overline{\rho}, \overline{u})\|_{H^{m-3}(\Omega)} \le C\varepsilon.$$

*Here, $(\overline{\rho}, \overline{u})$ is uniquely determined from*

$$(1.9) \qquad \overline{\rho} = (1 - \Delta)_{\mathrm{QG}}^{-1}\theta, \qquad \overline{u} = \nabla^\perp (1 - \Delta)_{\mathrm{QG}}^{-1}\theta,$$

*with $\theta$ solving the quasi-geostrophic equations*

$$(1.10) \qquad \begin{aligned} &\partial_t \theta + \overline{u}\cdot\nabla\theta = 0, \\ &\theta_0 = \rho_0 - \nabla^\perp \cdot u_0. \end{aligned}$$

*Here, the definition of $(1 - \Delta)_{\mathrm{QG}}^{-1}$ is given in section 5. It basically enforces a certain type of boundary condition on $(1 - \Delta)^{-1}$, suited for the quasi-geostrophic equations.*

*Proof* (sketch of proof). Section 5 is devoted to RSW equations. The key ingredients are the following. For (1.7), see Theorem 5.3, which is based on Lemmas 5.2 and 2.6. For (1.8)–(1.10), see Theorem 5.4, which is based on Lemmas 5.5 and 5.6. ☐

We note in passing that local-in-time existence of classical solutions to the incompressible Euler equations (1.5) (resp., the quasi-geostrophic equations (1.9), (1.10)) can be proved following techniques of [14] (resp., [4]). The boundary conditions in our systems do not require extra treatment as far as solution regularity is concerned.

The above theorems confirm that, with solid-wall boundary condition, the compressible fluid dynamics are approximated by their incompressible counterparts upon time averaging. Such an approximation is therefore in some sense of weak convergence. One of its theoretical implications is regarding the transport of passive scalars.[2]

COROLLARY 1.4. *Let $s(t, x)$ solve the linear transport equation*

$$\partial_t s + u\cdot\nabla s = 0,$$

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[2]For simplicity, we focus only on advective form (6.1). The conservative form $\partial_t s + \nabla\cdot(us) = 0$ can be treated similarly.

*subject to initial data $s_0(\cdot) \in H^m(\Omega)$. The velocity field $u$ is determined by the solution to the Euler equations as in Theorem* 1.1 *(resp., RSW equations as in Theorem* 1.3*). Then, by replacing $u$ with its incompressible counterpart $\overline{u}$ as defined in Theorem* 1.1 *(resp., Theorem* 1.3*),*

$$\partial_t \overline{s} + \overline{u} \cdot \nabla \overline{s} = 0, \qquad \overline{s}_0(\cdot) = s_0(\cdot),$$

*we establish that, for some finite time $T$,*

$$\max_{0 \leq t \leq T} \|s - \overline{s}\|_{H^{m-3}(\Omega)} \leq C\varepsilon.$$

The proof will be given in section 6.

There have been numerous results regarding the *singular limits* of compressible Euler equations and other fluid equations in various settings. We point to two survey papers for some comprehensive lists of references: Schochet [24] with emphases on hyperbolic PDEs and homogenization in space-time, and Masmoudi [18] with emphases on viscous fluids and weak solutions. To mention only a few earlier works, we note papers by Ebin [8, 9, 10], Beirão da Veiga [3], and Klainerman and Majda [15] for inviscid fluid equations, and papers by Kreiss [16] and Tadmor [25] for general hyperbolic PDEs. In a closely related paper [6], Browning, Kasahara, and Kreiss applied the bounded derivative method in numerical schemes to gain control on time derivatives and thus to get rid of fast gravity waves. These results, in terms of (1.2), confirmed that compressible flow $(\rho, u)$ converges to $(0, \overline{u})$ strongly at order $O(\varepsilon)$ with $\overline{u}$ solving (1.5) *provided* the initial data $(\rho_0, u_0)$ also converge to $(0, \overline{u}_0)$ strongly at order $O(\varepsilon)$:

(1.11) $$\|\rho_0\| + \|u_0 - \overline{u}_0\| \lesssim \varepsilon \quad \text{for some div-free } \overline{u}_0.$$

Here, $\|\cdot\|$ denotes some suitable spatial norm. Note that condition (1.11) implies that perturbation in the total density $\hat{\rho}$ vanishes at order $O(\varepsilon^2)$—consult (1.1).

This family of *well-prepared* initial data (1.11) directly implies uniform bound on the size of $\partial_t(\rho, u)$ at $t = 0$, independent of $\varepsilon$, by virtue of (1.2). Therefore, the so-called initial layer is suppressed. Then, one obtains uniform control on the size of $\partial_t(\rho, u)$ for finite times, which allows passing of limits by the Arzelà–Ascoli lemma. Well-prepared conditions on initial data were later removed for problems in the whole space (Ukai [27]), in an exterior domain (Isozaki [11, 12]), and in a torus (Schochet [23]). These arguments more or less rely on the use of Fourier analysis and/or the dispersive nature of the underlying wave equations.

Singular limit problems in a bounded spatial domain, on the other hand, remain much less studied. Schochet proved in [21] the same low-Mach-number limit with solid-wall boundary condition and, again, well-prepared initial data. A main challenge in this setting is the presence of characteristic boundary. It is elaborated in Rauch's work [19] for linear systems that, in general, only estimates along tangential directions are available near the boundary. We also note that there were also preceding results in, e.g., [10, 3], all of which required well-prepared initial data. In [20], Secchi proved the strong convergence of $u^P$ and weak* convergence of $u^Q$ for three-dimensional Euler equations with ill-prepared initial data. This result did not prove convergence rates or the case of RSW equations. Jones proved in [13] a convergence theorem for the RSW equations with well-prepared initial data and solid-wall boundary conditions.

The key argument of our paper is that, even with *ill-prepared* initial data and solid-wall boundary condition that entraps acoustic waves, nonlinear resonance of

fast acoustic waves does not enter the slow dynamics at all (Lemma 3.2) and interaction of fast-slow dynamics vanishes at order $O(\varepsilon)$ upon integrating in time (Lemma 3.3). Smallness condition (1.11) is no longer assumed, and thus the setting is more physically relevant. The key ideas used in this paper were partially originated in Cheng [7] for studying the RSW equations with two fast scales in the whole space. It recently came to our knowledge that similar approaches have occasionally appeared in the literature—cf. equations after (7) in Lions and Masmoudi [17] for weak limit problems, and cf. equations (4.27), (4.28) in Schochet [24] for problems without boundary. We also tackle the boundary condition carefully and present a clear calculation of a priori estimates independent of $\varepsilon$. Here, the vorticity equation plays a crucial role, which was argued in, e.g., Schochet [22]. Throughout our analysis (not just a priori estimates), we employ elliptic estimates for PDE systems with certain boundary conditions (Agmon, Douglis, and Nirenberg [1, 2]).

The organization of the rest of this article is as follows. Section 2 through 4 are devoted to the main ingredients listed in the sketched proof of Theorem 1.1 regarding the Euler equations. In particular, we establish a series of lemmas for a somewhat general family of hyperbolic PDEs. Next, in section 5, we study the RSW equations under the same framework as for the Euler's equations, but we focus more on the differences of these two systems, i.e., the elliptic operator with factor $1/\varepsilon$ and the associated projection operators. We then prove the main ingredients listed in the sketched proof of Theorem 1.3. In section 6, we show that the transportation properties of the compressible and incompressible flows differ only by $O(\varepsilon)$.

In more detail, in section 2, we introduce elliptic estimates particularly for the Euler equations and present a precise characterization for the projection operators associated with Helmholtz decomposition. These projection operators will be applied to decompose the solution as well as the system into "slow" and "fast" components. Moving on to section 3, we conduct a thorough study on nonlinear interactions of fast-fast and fast-slow types. In section 4, we give an elementary proof of $\varepsilon$-independent energy estimates that are necessary for all the convergence results to work.

We will repeatedly use some well-known inequalities of Sobolev norms without making references. They are all based on the Hölder inequality, the Gargliardo–Nirenberg inequality, and the Sobolev inequality. For the most part, it is sufficient to accept the following estimates:

$$(1.12) \qquad \|\partial_x^{j_1} g_1 \partial_x^{j_2} g_2 \ldots \partial_x^{j_k} g_k\|_{L^2(\Omega)} \leq c\|g_1\|_{H^m}\|g_2\|_{H^m} \ldots \|g_k\|_{H^m},$$

where $m > \mathbf{D}/2 + 1$, $0 \leq j_1 \leq \cdots \leq j_k \leq m$, and $j_1 + \cdots + j_k \leq m + 1$.

**2. Elliptic estimates and Helmholtz decomposition.** In section 2, 3, and 4, we study the Euler equations (1.2) with $F \equiv 0$. Rewrite it in a more compact form in terms of $U = \begin{pmatrix} \rho \\ u \end{pmatrix}$ as

$$(2.1) \qquad \partial_t U + \mathcal{N}(U, \nabla U; \varepsilon) = -\frac{1}{\varepsilon}\mathcal{L}[U], \quad u \cdot \mathsf{n}\big|_{\partial\Omega} = 0, \quad \int_\Omega \rho = 0,$$

where the nonlinear term

$$(2.2) \qquad \mathcal{N}(U_1, \nabla U_2; \varepsilon) := \begin{pmatrix} u_1 \cdot \nabla \rho_2 + \rho_1 \nabla \cdot u_2 \\ u_1 \cdot \nabla u_2 + g(\rho_1; \varepsilon)\nabla \rho_2 \end{pmatrix},$$

$$(2.3) \qquad \text{with} \quad g(\rho; \varepsilon) := \left(\frac{p'(1 + \varepsilon\rho)}{1 + \varepsilon\rho} - 1\right)\frac{1}{\varepsilon},$$

and the *elliptic operator* associated with the singular parameter $1/\varepsilon$

$$(2.4) \qquad \mathcal{L}\left[\begin{pmatrix} \rho \\ u \end{pmatrix}\right] := \begin{pmatrix} \nabla \cdot u \\ \nabla \rho \end{pmatrix}.$$

There is a "vorticity operator"

$$(2.5) \qquad \mathcal{K}\left[\begin{pmatrix} \rho \\ u \end{pmatrix}\right] := \nabla \times u$$

that cancels out $\mathcal{L}$, i.e.,

$$(2.6) \qquad \mathcal{KL} \equiv 0.$$

This is why the singular $1/\varepsilon$ term does not appear in the vorticity equations as follows:

$$(2.7) \qquad \partial_t(\nabla \times u) + \nabla \cdot (u \nabla \times u) = 0 \qquad \text{in two dimensions}$$

and

$$(2.8) \qquad \partial_t(\nabla \times u) + \nabla \cdot (u \nabla \times u) + (\nabla \times u) \cdot \nabla u = 0 \qquad \text{in three dimensions.}$$

The papers of Agmon, Douglis, and Nirenberg [1, 2] establish a complementing boundary condition that is necessary and sufficient for the solution operator of an $s$th order elliptic PDE system to be $\mathbf{C}^m \to \mathbf{C}^{m+s}$ and $H^m \to H^{m+s}$. To treat the Euler equations, only a particular case is used: for any velocity field $u$ with a trace subject to the solid-wall boundary condition $u \cdot \mathsf{n}\big|_{\partial\Omega} = 0$,

$$(2.9) \qquad \|u\|_{H^m(\Omega)} \leq C\left(\|\nabla \cdot u\|_{H^{m-1}(\Omega)} + \|\nabla \times u\|_{H^{m-1}(\Omega)} + \|u\|_{L^2(\Omega)}\right).$$

Here and below, we always assume $m$ is a positive integer so that the trace $u\big|_{\partial\Omega}$ is well defined. See, e.g., [5] for application of this estimate.

For the $\rho$ component of the solution, under the zero mean condition in (1.3), one has $\|\rho\|_{L^2(\Omega)} \leq C\|\nabla \rho\|_{L^2(\Omega)}$ by the Poincaré inequality and therefore

$$(2.10) \qquad \|\rho\|_{H^m(\Omega)} \leq C\|\nabla \rho\|_{H^{m-1}(\Omega)}.$$

Now, define a solution space $\mathbf{X}^m \subset H^m(\Omega)$ as

$$\mathbf{X}^m := \left\{ U = \begin{pmatrix} \rho \\ u \end{pmatrix} \in H^m(\Omega) \,\Big|\, u \cdot \mathsf{n} = 0 \text{ on } \partial\Omega \text{ and } \int_\Omega \rho\, dx = 0 \right\}.$$

Then, with elliptic operators defined in (2.4), (2.5), the above estimates (2.9), (2.10) lead to

$$(2.11) \quad \|U\|_{H^m(\Omega)} \leq C(\|\mathcal{L}[U]\|_{H^{m-1}(\Omega)} + \|\mathcal{K}[U]\|_{H^{m-1}(\Omega)} + \|u\|_{L^2(\Omega)}) \quad \text{for } U \in \mathbf{X}^m.$$

**2.1. Helmholtz decomposition.** Define $\mathcal{P}$ as the $L^2$ projection onto the $L^2$ closure of $\mathrm{Ker}\mathcal{L} \cap \mathbf{X}^m$, and define $\mathcal{Q}$ as its orthogonal complement,

$$(2.12) \qquad \mathcal{P} := L^2\text{–Proj}\{\overline{\mathrm{Ker}\mathcal{L} \cap \mathbf{X}^m}^{L^2}\}, \qquad \mathcal{Q} := \mathcal{I} - \mathcal{P}.$$

Note that $\mathrm{Ker}\mathcal{L} \cap \mathbf{X}^m = \left\{ \begin{pmatrix} 0 \\ u \end{pmatrix} \in H^m \,\big|\, u \cdot \mathsf{n}|_{\partial\Omega} = 0,\ \nabla \cdot u = 0 \right\}$.

These projections can be characterized conveniently by an elliptic PDE.

LEMMA 2.1. *Operators defined in* (2.12) *satisfy*

(2.13) $$\text{for any } U = \begin{pmatrix} \rho \\ u \end{pmatrix} \in H^m, \quad \mathcal{Q}[U] = \begin{pmatrix} \rho \\ \nabla\phi \end{pmatrix},$$

*with $\phi$ solving*

(2.14) $$\begin{cases} \Delta\phi = \nabla\cdot u, \\ \nabla\phi\cdot\mathsf{n}\big|_{\partial\Omega} = u\cdot\mathsf{n}\big|_{\partial\Omega}. \end{cases}$$

*Proof.* Solvability (unique module a constant) and regularity of (2.14) follow from standard elliptic PDE theory (see, e.g., [26, Chap. 5, Prop. 7.7]). It suffices to verify that $\mathcal{Q}$ defined in (2.13) is identical to $\mathcal{I} - \mathcal{P}$ defined in (2.12).

By definition,

$$\mathcal{L}(\mathcal{I} - \mathcal{Q})[U] = \mathcal{L}\left[\begin{pmatrix} 0 \\ u - \nabla\phi \end{pmatrix}\right] = \begin{pmatrix} 0 \\ \nabla\cdot u - \Delta\phi \end{pmatrix} = 0$$

$$\text{and} \quad \nabla(u - \nabla\phi)\cdot\mathsf{n}\big|_{\partial\Omega} = 0.$$

Therefore,

$$(\mathcal{I} - \mathcal{Q})[U] \in \text{Ker}\mathcal{L} \cap \mathbf{X}^m.$$

It remains to show that $\mathcal{Q}[U]$ is $L^2$-orthogonal to $\text{Img}\mathcal{P}$, i.e., $\text{Ker}\mathcal{L} \cap \mathbf{X}^m$. Take any function $U_1 = \begin{pmatrix} \rho_1 \\ u_1 \end{pmatrix} \in \text{Ker}\mathcal{L} \cap \mathbf{X}^m$ so that $\rho_1 = 0$ and $\nabla\cdot u_1 = 0$ with $u_1\cdot\mathsf{n}\big|_{\partial\Omega} = 0$. Then,

$$\int_\Omega U_1\cdot\mathcal{Q}[U]\,dx = \int_\Omega \rho_1\rho + u_1\cdot\nabla\phi\,dx = \int_\Omega \nabla\cdot(u_1\phi)\,dx = \int_{\partial\Omega} \mathsf{n}\cdot u_1\phi\,ds = 0.$$

Thus, $\mathcal{Q}[U]$ is $L^2$-orthogonal to $\text{Ker}\mathcal{L} \cap \mathbf{X}^m$ and therefore its $L^2$ closure.  □

Now that $\mathcal{P}$, $\mathcal{Q}$ are well defined, for simplicity, we will use $U^P$ for $\mathcal{P}[U]$ and $U^Q$ for $\mathcal{Q}[U]$ whenever it is not ambiguous. We will also use lowercase $u^P$, $u^Q$ for the associated velocity components. The density component of $\mathcal{P}[U]$ is always zero.

The following property of $\mathcal{P}$ will be a key element in estimating nonlinear resonances of acoustic waves. For example, it is necessary to have $\mathcal{P}\mathcal{L}\big|_{\mathbf{X}^{m+1}} = 0$ so that applying $\mathcal{P}$ on (2.1) eliminates the $1/\varepsilon$ term.

LEMMA 2.2. *A duality relation holds true,*

(2.15) $$\text{Ker}\mathcal{P}\Big|_{H^m} = \text{Img}\mathcal{L}\Big|_{\mathbf{X}^{m+1}},$$

(2.16) $$\text{Img}\mathcal{Q}\Big|_{H^m} = \text{Img}\mathcal{L}\Big|_{\mathbf{X}^{m+1}}.$$

*Proof.* Since

$$U \in \text{Ker}\mathcal{P} \iff \mathcal{P}[U] = 0 \iff \mathcal{Q}[U] = U \iff U \in \text{Img}\mathcal{Q},$$

it suffices to prove (2.16).

Take any $U^Q = \begin{pmatrix} \rho \\ u^Q \end{pmatrix} \in H^m$. By (2.13), (2.14), $u^Q = \nabla\phi$ for some $\phi \in H^{m+1}$ with $\int_\Omega \phi = 0$. Also, since $\int_\Omega \rho = 0$, the Poisson equation $\Delta\psi = \rho$ with $\nabla\psi\cdot\mathsf{n}\big|_{\partial\Omega} = 0$

admits (at least) a solution in $H^{m+2}$. Thus, we set $U_1 = \begin{pmatrix} \phi \\ \nabla\psi \end{pmatrix} \in \mathbf{X}^{m+1}$ and clearly $U^Q = \mathcal{L}[U_1]$. Therefore, LHS(2.16) $\subset$ RHS(2.16).

Assume $U = \begin{pmatrix} \rho \\ u \end{pmatrix} = \mathcal{L}[U_2]$ for some $U_2 = \begin{pmatrix} \rho_2 \\ u_2 \end{pmatrix} \in \mathbf{X}^{m+1}$. Since $u = \nabla\rho_2$, by (2.13), (2.14) and its unique solvability, we have $u^Q = u$. Thus, $U = U^Q \in \text{Img}\mathcal{Q}$. Therefore, RHS(2.16) $\subset$ LHS(2.16). $\qquad\square$

REMARK 2.3. *The above duality relation has an analogue in linear algebra: if $\mathcal{L}$ is identified with a (skew)-symmetric $n \times n$ matrix, then $\text{Ker}\mathcal{P} = \text{Img}\mathcal{L}$. What's more, if $\mathcal{KL} = 0$ and $\text{rank}\mathcal{K} + \text{rank}\mathcal{L} = n$, then $\text{Ker}\mathcal{K} = \text{Img}\mathcal{L}$. In fact, in our case, if $\Omega$ is contractible, one can use the Poincaré lemma to show that indeed $\text{Ker}\mathcal{K} = \text{Img}\mathcal{L}$ in proper regularity spaces.*

**2.2. Boundedness of projection operators.** We claim that $\mathcal{P}$, $\mathcal{Q}$ are both bounded operators in $\mathbf{X}^m$. Indeed, definitions (2.12)–(2.14) lead to the Pythagorean theorem $\|u\|_{L^2} = \|u^P\|_{L^2} + \|u^Q\|_{L^2}$. Then, apply elliptic estimate (2.11) on $\mathcal{P}[U]$ together with $\mathcal{LP} = 0$ (by definition) as well as $\mathcal{KP} = \mathcal{K}$ (since Lemma 2.2 $\implies$ $\text{Img}\mathcal{Q} = \text{Img}\mathcal{L} \subset \text{Ker}\mathcal{K} \implies \mathcal{KQ} = 0$) to obtain

$$(2.17) \qquad \|\mathcal{P}[U]\|_{H^m} \leq C(\|\mathcal{K}[U]\|_{H^{m-1}} + \|u^P\|_{L^2}) \leq C'\|U\|_{H^m} \quad \text{for } U \in H^m.$$

Thanks to duality,

$$(2.18) \qquad \|\mathcal{Q}[U]\|_{H^m} \leq C(\|\mathcal{L}[U]\|_{H^{m-1}} + \|u^Q\|_{L^2}) \leq C'\|U\|_{H^m} \quad \text{for } U \in \mathbf{X}^m.$$

But we will need a stronger version of (2.18) which is stated and proved in Lemma 2.4. A heuristic argument is the following: in the linear algebra setting given in Remark 2.3, the restriction of $\mathcal{L}$ onto the image of $\mathcal{Q}$ is invertible and thus $\|U^Q\| \leq C\|\mathcal{L}[U^Q]\|$ in proper norms. In the same spirit, for our elliptic operators, we have the following lemma.

LEMMA 2.4. *For any $U \in \mathbf{X}^m$ ($m \geq 1$),*

$$\|\mathcal{Q}[U]\|_{H^m} \leq C\|\mathcal{L}[U]\|_{H^{m-1}}.$$

*Proof.* The first part of (2.18) is established; it remains to estimate $\|u^Q\|_{L^2}$.

By Lemma 2.2, $u^Q = \nabla\phi$ with $\phi$ chosen to have zero mean. By definition (2.13), (2.14) and $U \in \mathbf{X}^m$,

$$u^Q \cdot \mathsf{n}\big|_{\partial\Omega} = u \cdot \mathsf{n}\big|_{\partial\Omega} = 0,$$

and thus

$$\int_\Omega |u^Q|^2 = \int_\Omega u^Q \cdot \nabla\phi = \int_\Omega \nabla \cdot u^Q \phi$$
$$\leq \|\nabla \cdot u^Q\|_{L^2}\|\phi\|_{L^2} \leq C\|\nabla \cdot u^Q\|_{L^2}\|\nabla\phi\|_{L^2}.$$

Since $u^Q = \nabla\phi$, we deduce $\|u^Q\|_{L^2} \leq C\|\nabla \cdot u^Q\|_{L^2}$, which obviously is bounded by $C\|\mathcal{L}[U^Q]\|_{H^{m-1}}$. $\qquad\square$

This estimate immediately leads to the proof of (1.6) regarding the time average of $U^Q$, the fast acoustic component of the dynamics.

THEOREM 2.5. *Consider the Euler equations (2.1)–(2.4) under the same assumptions as in Theorem 1.1. Then, (1.6) holds true, that is,*

$$\left\|\frac{1}{t}\int_0^t U^Q d\tau\right\|_{H^m(\Omega)} \leq C\varepsilon\left(1 + \frac{1}{t}\right) \qquad for\ t \in (0, T].$$

*Here, C depends only on* $\max_{t\in[0,T]} \|U\|_{H^m(\Omega)}$ *and* $m$, *and it is independent of* $\varepsilon$ *and $t$ otherwise.*

The proof is a straightforward application of Lemma 2.4 and the following lemma.

LEMMA 2.6. *Consider time-dependent equation*

$$U_t = \frac{1}{\varepsilon}\mathcal{L}[U] + f(t,q)$$

*over certain spatial domain* $\Omega$. *Here,* $\varepsilon > 0$ *is a scaling constant,* $f(t,q)$ *a source term, and* $\mathcal{L}$ *a linear operator independent of time. Let operator* $\mathcal{P}$ *denote (some) projection onto the null space of* $\mathcal{L}$. *Assume a priori* $U, f(t,q), \mathcal{L}[U], \mathcal{P}[U]$ *have enough regularity as needed.*

*Then, under the assumption*

(2.19)                    $$\|U - \mathcal{P}[U]\|_{H^{k_1}(\Omega)} \leq C\|\mathcal{L}[U]\|_{H^{k_2}(\Omega)}$$

*for some constant* $C$, *the following estimate on the* time average *of* $U$ *holds true:*

$$\left\| \frac{1}{T}\int_0^T U\,dt - \frac{1}{T}\int_0^T \mathcal{P}[U]\,dt \right\|_{H^{k_1}} \leq \varepsilon C\left(\frac{2M}{T} + M'\right),$$

*where constants* $M := \max_{t\in[0,T]} \|U(t,\cdot)\|_{H^{k_2}}$ *and* $M' := \max_{t\in[0,T]} \|f(t,\cdot)\|_{H^{k_2}}$.

*Proof.* First, transform the original equation into

$$U_t = \frac{1}{\varepsilon}\mathcal{L}[U - \mathcal{P}[U]] + f$$

and apply time averaging $\frac{1}{T}\int_0^T \cdot\,dt$ on both sides:

$$\frac{1}{T}(U(T,\cdot) - U(0,\cdot)) = \frac{1}{\varepsilon T}\int_0^T \mathcal{L}[U - \mathcal{P}[U]]dt + \frac{1}{T}\int_0^T f(t,\cdot)dt.$$

Since all necessary regularities were assumed available and $\mathcal{L}$ was assumed to be linear and independent of time, we argue that $\int_0^T \cdot\,dt$ and $\mathcal{L}[\cdot]$ commute so that the above equation becomes

$$\frac{1}{T}(U(T,\cdot) - U(0,\cdot)) = \frac{1}{\varepsilon}\mathcal{L}\left[\frac{1}{T}\int_0^T U\,dt - \frac{1}{T}\int_0^T \mathcal{P}[U]\,dt\right] + \frac{1}{T}\int_0^T f(t,\cdot)\,dt.$$

Due to the factor $\frac{1}{\varepsilon}$ in the first term on the right-hand side (RHS), we have

$$\left\| \mathcal{L}\left[\frac{1}{T}\int_0^T U\,dt - \frac{1}{T}\int_0^T \mathcal{P}[U]\,dt\right] \right\|_{H^{k_2}} \leq \varepsilon\left(\frac{2M}{T} + M'\right).$$

Finally, apply estimate (2.19) to arrive at the conclusion.  □

**3. Estimates on nonlinear interaction and strong convergence.** In this section, we always assume the solution to the above system exists and satisfies $U \in \mathbf{X}^m$ for some $m > \mathbf{D}/2 + 4$. It then follows that $\partial_t U \in \mathbf{X}^{m-1}$, $\int U\,dt \in \mathbf{X}^m$, etc.

THEOREM 3.1. *Consider the* $\mathbf{D}$-*dimensional barotropic compressible Euler equations* (2.1). *Assume a solution exists classically:* $U(t,x) \in \cap_{j=0}^m \mathbf{C}^j([0,T]; H^{m-j}(\Omega))$ *with* $m > \mathbf{D}/2 + 4$. *Also assume a nonvacuum condition* $\|\varepsilon\rho\|_{\mathbf{C}([0,T];H^m)} \leq 1/2$. *Then,*

*with projection $\mathcal{P}$ defined in (2.12), there exists an incompressible flow $\overline{U} = \left(\begin{smallmatrix} 0 \\ \overline{u} \end{smallmatrix}\right)$ so that*

$$\max_{t \in [0,T]} \|\mathcal{P}[U] - \overline{U}\|_{m-3} \leq \varepsilon C.$$

*Here, $C$ depends only on $\max_{t \in [0,T]} \|U\|_{H^m(\Omega)}$, $m$, and $T$, and it is independent of $\varepsilon$.*

*In particular, $\overline{u}$ can be chosen as the unique $H^m$ solution to the incompressible Euler equations*

$$(3.1) \qquad \begin{cases} \partial_t \overline{u} + \overline{u} \cdot \nabla \overline{u} + \nabla \overline{q} = 0, \\ \qquad\qquad\qquad \nabla \cdot \overline{u} = 0, \end{cases}$$
$$\overline{u} \cdot \mathsf{n}\big|_{\partial\Omega} = 0, \quad \overline{u}(0, \cdot) = u_0^P.$$

*In other words, $\overline{U}$ solves*

$$(3.2) \qquad \partial_t \overline{U} + \mathcal{P}\mathcal{N}(\overline{U}, \nabla\overline{U}; 0) = 0.$$

The key to proving this theorem is contained in two lemmas. They have been used in [7] for the RSW equations without boundary constraints. We now migrate them to bounded domains using the properly defined projection operators $\mathcal{P}$, $\mathcal{Q}$.

Apply $\mathcal{P}$ to (2.1). By Lemma 2.2, $\mathcal{P}\mathcal{L}\big|_{\mathbf{X}^{m+1}} = 0$. Thus,

$$(3.3) \qquad \partial_t U^P + \mathcal{P}\mathcal{N}(U, \nabla U; \varepsilon) = 0.$$

Although $\mathcal{N}$ defined in (2.2) is not entirely bilinear, we take advantage of the fact that $p'(1) = 1$ and $\rho^P = 0$ to rewrite

$$(3.4) \qquad \begin{aligned} &\partial_t U^P + \mathcal{P}\begin{pmatrix} 0 \\ u^P \cdot \nabla u^P \end{pmatrix} + \mathcal{P}\mathcal{N}(U^Q, \nabla U^Q; \varepsilon) \\ &\quad + \mathcal{P}\begin{pmatrix} \nabla \cdot (u^P \rho^Q) \\ u^P \cdot \nabla u^Q + u^Q \cdot \nabla u^P \end{pmatrix} = 0. \end{aligned}$$

In comparison with (3.2), there are two types of nonlinear interactions to be studied,

$$\text{``fast-fast''} \qquad \mathcal{P}\mathcal{N}(U^Q, U^Q; \varepsilon),$$
$$\text{``fast-slow''} \qquad \mathcal{P}\begin{pmatrix} \nabla \cdot (u^P \rho^Q) \\ u^P \cdot \nabla u^Q + u^Q \cdot \nabla u^P \end{pmatrix}.$$

At first glance, one expects the nonlinearity of $\mathcal{N}$ to generate resonance from all kinds of interaction between $U^P$ and $U^Q$. However, the following lemma *excludes* the contribution of "fast-fast" interaction from $\mathrm{Img}\mathcal{P}$.

LEMMA 3.2. *For any $U^Q \in \mathrm{Img}\mathcal{Q}$ with sufficient regularity,*

$$\mathcal{P}\mathcal{N}(U^Q, \nabla U^Q; \varepsilon) = 0.$$

*Proof.* The density component of $\mathrm{Img}\mathcal{P}$ is always zero. By Lemma 2.2, it suffices to show that

$$u^Q = \nabla\phi \quad \text{implies} \quad u^Q \cdot \nabla u^Q + g(\rho; \varepsilon)\nabla\rho \text{ is also a gradient.}$$

This is true due to the calculus identity

$$(\nabla\phi)\cdot\nabla(\nabla\phi) = \frac{1}{2}\nabla|\nabla\phi|^2$$

and the fact $g(\rho;\varepsilon)$ is a function of $\rho$ only. $\square$

Next, move on to the "fast-slow" interaction: $\mathcal{P}\left(\begin{smallmatrix} \nabla\cdot(u^P\rho^Q) \\ u^P\cdot\nabla u^Q + u^Q\cdot\nabla u^P \end{smallmatrix}\right)$. Although it does not have the same cancellation property as the "fast-fast" interaction, the next lemma reveals that, upon *averaging in time*, it vanishes at order $O(\varepsilon)$. In other words, the "fast-slow" interaction is asymptotically negligible when averaged in time.

LEMMA 3.3. *For any* $U = U^Q + U^P \in \mathbf{X}^m$ *with* $m > \mathbf{D}/2 + 3$ *that solves* (2.1),

$$\left\| \int_0^t \mathcal{P}\left(\begin{matrix} \nabla\cdot(u^P\rho^Q) \\ u^P\cdot\nabla u^Q + u^Q\cdot\nabla u^P \end{matrix}\right) d\tau \right\|_{H^{m-2}} \le \varepsilon C(t+1)^2 \quad .$$

*Here,* $C$ *depends only on* $\max_{t\in[0,T]} \|U\|_{H^m(\Omega)}$ *and* $m$, *and it is independent of* $\varepsilon$ *and* $t$ *otherwise.*

*Proof.* Since the definition of $\mathcal{P}$ is independent of the density component, it suffices to estimate

$$\int_0^T \mathcal{P}\left[\left(\begin{matrix} 0 \\ u^P\cdot\nabla u^Q + u^Q\cdot\nabla u^P \end{matrix}\right)\right] dt.$$

Due to the assumption of $H^m$ regularity with $m > \mathbf{D}/2 + 3$, we have $U, U^P, U^Q$ in $C^2$ and therefore, by repeatedly applying the Fubini theorem, the time integral $\int \cdot\, dt$ and operator $\mathcal{P}$ above can be switched:

$$\int_0^T \mathcal{P}\left[\left(\begin{matrix} 0 \\ u^P\cdot\nabla u^Q + u^Q\cdot\nabla u^P \end{matrix}\right)\right] dt = \mathcal{P}\left[\int_0^T \left(\begin{matrix} 0 \\ u^P\cdot\nabla u^Q + u^Q\cdot\nabla u^P \end{matrix}\right) dt\right].$$

Then, by (2.17), which is not restricted by the solid-wall boundary condition, $\mathcal{P}$ can be dropped from the estimate,

$$\left\| \int_0^T \mathcal{P}\left[\left(\begin{matrix} 0 \\ u^P\cdot\nabla u^Q + u^Q\cdot\nabla u^P \end{matrix}\right)\right] dt \right\|_{H^{m-2}} \le C \left\| \int_0^T \left(\begin{matrix} 0 \\ u^P\cdot\nabla u^Q + u^Q\cdot\nabla u^P \end{matrix}\right) dt \right\|_{H^{m-2}}.$$

To illustrate the steps in estimating the RHS above, we explain in details on $u^Q\cdot\nabla u^P$, the other part being very similar.

The key idea is to utilize the time average of $u^Q$ which has been estimated in Theorem 2.5,

$$\|w(t,\cdot)\|_{H^m(\Omega)} \le \varepsilon C(t+1),$$

where we define

$$w(t,\cdot) := \int_0^t u^Q(\tau,\cdot)d\tau.$$

Now, integrate by parts in time,

$$\int_0^t u^Q\cdot\nabla u^P\, d\tau = w\cdot\nabla u^P\Big|_0^t - \int_0^t w\cdot\nabla\partial_t u^P\, d\tau.$$

Then, estimate the time derivative of the slow dynamics $\partial_t u^P$ by using (3.3) and the fact that $\mathcal{N}$ depends on $\varepsilon$ via positive powers of $\varepsilon\rho$,

$$\|\partial_t u^P\|_{H^{m-1}} \leq C.$$

Therefore, we combine the above four (in)equalities to conclude that

$$\left\|\int_0^t u^Q \cdot \nabla u^P \, dt\right\|_{H^{m-2}} \leq \left\|w \cdot \nabla u^P\Big|_0^t\right\|_{H^{m-2}} + \int_0^t \left\|w \cdot \nabla \partial_t u^P \, d\tau\right\|_{H^{m-2}} d\tau$$
$$\leq \varepsilon C(t+1)^2. \quad \square$$

To close this section, we prove Theorem 3.1.

*Proof of Theorem* 3.1. In this proof, we assume the existence time of classical solutions to the compressible and incompressible Euler equations depends solely on the size of initial data and is otherwise independent of $\varepsilon$. For a detailed proof on uniform energy estimates, consult section 4.

Lemmas 3.2 and 3.3 together with (3.4) lead to

$$(3.5) \qquad \text{for } -\xi(t, \cdot) := \int_0^t \partial_t u^P + \mathcal{P}[u^P \cdot \nabla u^P] \, d\tau,$$

$$(3.6) \qquad \|\xi\|_{H^{m-2}(\Omega)} \leq \varepsilon C(t+1)^2.$$

Here, one can regard $-\xi$ as the time integral of the "residual" that results from approximating $u^P$ using the incompressible Euler equations (3.1). Thus, we subtract the time derivative of (3.5) from (3.1) to obtain an equation

$$(3.7) \qquad \partial_t \xi = \partial_t \delta + \mathcal{P}\left[\overline{u} \cdot \nabla \delta + \delta \cdot \nabla u^P\right],$$

where we define an error term

$$\delta := \overline{u} - u^P.$$

Take spatial derivative $\partial^\alpha$ with $|\alpha| \leq m - 3$ on (3.7), and obtain

$$(3.8) \qquad \partial_t \xi^\alpha = \partial_t \delta^\alpha + \mathcal{P}\left[\overline{u} \cdot \nabla \delta^\alpha\right] + f^\alpha,$$
$$\text{where} \quad \delta^\alpha := \partial^\alpha \delta, \ \xi^\alpha := \partial^\alpha \xi$$
$$\text{and} \quad f^\alpha := \mathcal{P}\left[\partial^\alpha\left(\overline{u} \cdot \nabla \delta\right) - \overline{u} \cdot \nabla(\partial^\alpha \delta)\right] + \mathcal{P}\left[\partial^\alpha(\delta \cdot \nabla u^P)\right].$$

Since $|\alpha| \leq m - 3$, by the boundedness of $\mathcal{P}$ in (2.17) and calculus inequalities (1.12), we have

$$(3.9) \qquad \|f^\alpha\|_{L^2} \leq C\|\delta\|_{H^{m-3}}.$$

Note that, in (3.8), the $\partial_t \xi$ is an $O(1)$ term. So we further need to further rewrite it as

$$(3.10) \qquad 0 = \partial_t(\delta^\alpha - \xi^\alpha) + \mathcal{P}\left[\overline{u} \cdot \nabla(\delta^\alpha - \xi^\alpha)\right] + f_1^\alpha + f^\alpha,$$

where

$$f_1^\alpha := \mathcal{P}\left[\overline{u} \cdot \nabla \xi^\alpha\right],$$

and by (3.6) and also $|\alpha| \leq m - 3$, we estimate

$$(3.11) \qquad\qquad \|f_1^\alpha\|_{L^2} \leq C\varepsilon.$$

Now, we are ready to perform energy estimates. Take the $L^2(\Omega)$-inner product of (3.10) with $\delta^\alpha - \xi^\alpha$, and use the fact that $\delta^\alpha - \xi^\alpha$ is div-free and therefore $L^2$-orthogonal to $\mathrm{Img}\mathcal{Q}$:

$$0 = \frac{1}{2}\partial_t\Big\langle \delta^\alpha - \xi^\alpha,\, \delta^\alpha - \xi^\alpha \Big\rangle + \Big\langle \overline{u}\cdot\nabla(\delta^\alpha - \xi^\alpha),\, (\delta^\alpha - \xi^\alpha) \Big\rangle$$
$$+ \Big\langle f_1^\alpha + f^\alpha,\, \delta^\alpha - \xi^\alpha \Big\rangle.$$

Then, apply the Stokes theorem to the second term on the RHS,

$$\Big\langle \overline{u}\cdot\nabla(\delta^\alpha - \xi^\alpha),\, (\delta^\alpha - \xi^\alpha) \Big\rangle_{L^2(\Omega)} = \frac{1}{2}\int_{\partial\Omega} (\overline{u}\cdot\mathsf{n})|\delta^\alpha - \xi^\alpha|^2 = 0,$$

where the last equality is due to the solid-wall boundary condition.

Combining the above two equalities with estimates (3.9), (3.11), we arrive at

$$\frac{1}{2}\partial_t\langle \delta^\alpha - \xi^\alpha, \delta^\alpha - \xi^\alpha \rangle \leq C\|\delta^\alpha - \xi^\alpha\|_{L^2}(\|\delta\|_{H^{m-3}} + \varepsilon).$$

Sum all such inequalities over all $\alpha$ with $|\alpha| \leq m - 3$ to obtain

$$\frac{1}{2}\partial_t\|\delta - \xi\|_{H^{m-3}}^2 \leq C\|\delta - \xi\|_{H^{m-3}}(\|\delta\|_{H^{m-3}} + \varepsilon).$$

Finally, solve this Gronwall inequality with initial conditions $\delta(0,\cdot) = \xi(0,\cdot) \equiv 0$ and estimate (3.6) to complete the proof.   $\square$

**4. A priori energy estimates, independent of $\varepsilon$.** Recall the equation of state, pressure $= p(1 + \varepsilon\rho)$ with $p(1) = 0$, $p'(1) = 1$, and introduce a new unknown $r := \frac{p(1+\varepsilon\rho)}{\varepsilon}$ so that the Euler equations (1.2) with $F \equiv 0$ (resp., the RSW equations with $F = -u^\perp$) are reformulated as

$$(4.1) \qquad \begin{cases} \dfrac{1}{\left[p^{-1}(\varepsilon r)\right]^2 p'(p^{-1}(\varepsilon r))}\,(\partial_t r + u\cdot\nabla r) + \dfrac{1}{p^{-1}(\varepsilon r)}\dfrac{\nabla\cdot u}{\varepsilon} = 0, \\[4mm] \partial_t u + u\cdot\nabla u + \dfrac{1}{p^{-1}(\varepsilon r)}\dfrac{\nabla r}{\varepsilon} = \dfrac{1}{\varepsilon}F. \end{cases}$$

Then, rescale this system by replacing

$$(4.2) \qquad u = \breve{u}/\varepsilon, \quad r = \breve{r}/\varepsilon, \quad t = \breve{t}\varepsilon, \quad F = \breve{F}/\varepsilon,$$

and arrive at

$$\begin{cases} \dfrac{1}{\left[p^{-1}(\breve{r})\right]^2 p'(p^{-1}(\breve{r}))}\,(\partial_{\breve{t}}\breve{r} + \breve{u}\cdot\nabla\breve{r}) + \dfrac{1}{p^{-1}(\breve{r})}\nabla\cdot\breve{u} = 0, \\[4mm] \partial_{\breve{t}}\breve{u} + \breve{u}\cdot\nabla\breve{u} + \dfrac{1}{p^{-1}(\breve{r})}\nabla\breve{r} = \breve{F}. \end{cases}$$

Finally, rewrite it as a symmetric hyperbolic PDE system for the rescaled variable $V := \begin{pmatrix} \breve{r} \\ \breve{u} \end{pmatrix}$,

$$(4.3) \qquad A_0(V)\partial_t V + \sum_{j=1}^{\mathbf{D}} A_j(V)\partial_{x_j} V + \mathcal{L}[V] = 0,$$

with the obvious definitions of $A_0$, $A_j$ which are all symmetric matrices. In particular, $A_1, \ldots, A_{\mathbf{D}}$ and first spatial/time derivatives of $A_0$ vanish at order $O(\|V\|_{H^m})$ about $V \equiv 0$.

This rescaled system will be the main subject of this section, for it offers the convenience of being free of any $\varepsilon$ terms.

THEOREM 4.1. *Let $m > \mathbf{D}/2 + 1$. There exist $\varepsilon$-independent constants $C^*$, $C^{**}$, $C^{***}$ dependent only on $m, \Omega$, and the pressure law so that the rescaled compressible Euler equations (4.3) (resp., the rescaled RSW equations) with initial data $V_0$ satisfying the compatibility condition (cf. Remark 1.2) admit a unique solution in the class of*

$$\sum_{j=0}^{m} \mathbf{C}^j \left( \left[ 0, \frac{C^*}{\|V_0\|_{H^m}} \right] ; H^{m-j}(\Omega) \right) \quad \textit{if} \quad \|V_0\|_{H^m} \le C^{**}.$$

*The solution is uniformly bounded $\|V\|_{H^m(\Omega)} \le C^{***}\|V_0\|_{H^m}$ on this finite time interval.*

*Consequently, the original compressible Euler equations (4.1) (resp., the RSW equations) with $\varepsilon \ll 1$ and initial data $(r_0, u_0)$ satisfying the compatibility condition admit a unique solution in the class of*

$$\sum_{j=0}^{m} \mathbf{C}^j \left( \left[ 0, \frac{C^*}{\|(r_0, u_0)\|_{H^m}} \right] ; H^{m-j}(\Omega) \right) \quad \textit{if} \quad \|(r_0, u_0)\|_{H^m} \le C^{**}/\varepsilon.$$

*The solution is uniformly bounded $\|(r, u)\|_{H^m(\Omega)} \le C^{***}\|(r_0, u_0)\|_{H^m}$ on this finite time interval.*

*Proof.* Let's first explain the key parts of the proof.

First, short-time existence of classical solutions is established in Rauch [19] for linear systems and in Schochet [21, 22] for certain nonlinear systems including our (1.2). To prove an $\varepsilon$-independent life span of $H^m$ solutions, it suffices to establish $\varepsilon$-independent a priori estimates and then simply apply the continuation principle.

To this end, for the $1/\varepsilon$ terms to vanish in the estimates, we start with the vortical component $\mathcal{K}[V]$ based on the $\varepsilon$-free vorticity equation. We also estimate the time derivatives using the skew-self-adjointness of $\frac{1}{\varepsilon}\mathcal{L}$ w.r.t. the still-valid boundary condition. Estimation of the acoustic part $\mathcal{L}[V]$ is based on the hyperbolic PDE system itself. The details of this framework are given in the proof of Theorem 4.2. We now apply the result of Theorem 4.2 to the nonlinear system (4.3).

Set $B_j = A_j(V)$ ($j = 0, 1, \ldots, \mathbf{D}$) in Theorem 4.2. All conditions are satisfied, especially the structural assumption (4.8) due to the existence of an actual vorticity equation for the compressible Euler equations. Next, replace all the time derivatives in (4.10) with spatial derivatives based on the system itself (4.3). Then, the definitions of $A_j$, calculus inequalities (1.12), and the fact that zeroth order derivative of $B_0 = A_0$ is excluded from estimate (4.10) imply that all the nonlinear products in (4.10) vanish at

order $O(\|V\|_{H^m}^2)$ around $V \equiv 0$. Therefore, there exist two smooth functions $g(\cdot), h(\cdot)$ that are strictly increasing and vanish at 0 so that, for $a(t) := \|V\|_{H^m_x}(t)$,

$$(4.4) \qquad a(t) \leq h(a_0) + ag(a) + \int_0^t ag(a)\, d\tau.$$

By assumptions on $h$, $g$, we have that $g \circ h$ is an automorphism of $[0, \infty]$. Thus, let $\gamma$ be the only positive root of

$$g \circ h(\gamma) = 0.5,$$

and let $T$ be the latest time so that

$$a(t) \leq h(\gamma) \quad \text{for } t \in [0, T].$$

Then, in such a setting $g(a(t)) \leq g(h(\gamma)) = 0.5$ for $t \in [0, T]$, and (4.4) implies

$$a(t) \leq h(a_0) + \frac{1}{2}a + \int_0^t \frac{1}{2}a\, d\tau \qquad \text{for } t \in [0, T],$$

$$\text{i.e.,} \qquad a(t) \leq 2h(a_0) + \int_0^t a\, d\tau \qquad \text{for } t \in [0, T].$$

This Gronwall's inequality implies

$$a(t) \leq 2h(a_0)e^t \qquad \text{for } t \in [0, T].$$

Finally, the theorem is proved by setting, e.g.,

$$C^* = \ln 2, \quad C^{**} = h^{-1}\left(\frac{\gamma}{4}\right), \quad C^{***} = \max_{a_0 \in [0, C^{**}]} \frac{h(a_0)}{a_0}. \qquad \Box$$

Our Theorems 4.1 and 4.2 and their proofs provide clear evidence that the continuation principle can be performed on a uniform time interval independent of $\varepsilon$. The uniform estimates still remain valid even after, as a standard procedure, we add a vanishing $\varepsilon_1 \mathsf{n} \cdot \nabla u$ term to the system to create a nonsingular boundary matrix. The reasons are the following: 1. the boundary matrix is still dissipative; 2. a vorticity equation still exists.

We also note in passing that a crucial "maximal positivity" condition on the boundary matrix and boundary condition is needed for all these existence theories to work (cf. Rauch [19] and Schochet [22]), but our system (4.1) is a canonical case satisfying this condition for sufficiently small $\varepsilon$.

THEOREM 4.2. *Consider the linear symmetric hyperbolic system*

$$(4.5) \qquad B_0 \partial_t V + \sum_{j=1}^{\mathbf{D}} B_j \partial_{x_j} V + \mathcal{L}[V] = 0,$$

$$(4.6) \qquad \mathcal{M}[u]\big|_{\partial\Omega} = 0,$$

*with all the eigenvalues of $B_0$ located in $[1/2, 2]$ and $\mathcal{M}$ being a time-independent operator defined on $\partial\Omega$. Assume $\mathcal{L}$ is nonnegative w.r.t. the boundary condition above:*

$$(4.7) \qquad \mathcal{M}[u]\big|_{\partial\Omega} = 0 \implies \langle V, \mathcal{L}[V] \rangle_{L^2_x} \geq 0.$$

*In addition, impose a structural assumption that there exists a "vorticity" operator $\mathcal{K}$ of first order differentiation so that applying $\mathcal{K}$ to this system results in a "vorticity equation"*

$$(4.8) \qquad \partial_t \mathcal{K}[V] + \sum_{j=1}^{\mathbf{D}} \partial_{x_j}(\tilde{B}_j \mathcal{K}[V]) = 0.$$

*Assume both $\{B_j\}$ and $\{\tilde{B}_j\}$ are symmetric matrices; also assume they are conservative or dissipative on $\partial\Omega$:*

$$(4.9) \qquad B_{\mathsf{n}}\big|_{\partial\Omega} \geq 0, \qquad \tilde{B}_{\mathsf{n}}\big|_{\partial\Omega} \geq 0,$$

*where the boundary matrix is defined as $B_{\mathsf{n}}\big|_{\partial\Omega} := \sum_{j=1}^{\mathbf{D}} \nu_j B_j$ for $\{\nu_j\}$ being the coordinates of the outward norm $\mathsf{n}$ on $\partial\Omega$.*

*Then, for system (4.5)–(4.9) with solution $V \in \sum_{j=0}^m \mathbf{C}^j([0,T]; H^{m-j}(\Omega))$ for $m > \mathbf{D}/2 + 1$, we have estimates*

(4.10)

$$\|V(t,\cdot)\|_{H_x^m} \lesssim \sum_{n=0}^{m} \|\partial_t^n V(0,\cdot)\|_{H_x^{m-n}}$$

$$+ \sum_{(j,\alpha,\beta)\in\mathbf{A}_m} \left\|(\partial_{t,x}^\alpha B_j)(\partial_{t,x}^\beta V)\right\|_{L_x^2} + \left\|(\partial_{t,x}^\alpha \tilde{B}_j)(\partial_{t,x}^\beta V)\right\|_{L_x^2}$$

$$+ \sum_{(j,\alpha,\beta)\in\mathbf{A}_m} \int_0^t \left\|(\partial_{t,x}^\alpha B_j)(\partial_{t,x}^\beta V)\right\|_{L_x^2} + \left\|(\partial_{t,x}^\alpha \tilde{B}_j)(\partial_{t,x}^\beta V)\right\|_{L_x^2} d\tau =: \mathsf{S},$$

*where*

$$\mathbf{A}_m := \left\{ (j,\alpha,\beta) \,\Big|\, j \in [0,m], \, |\alpha| + j > 0, \, |\alpha + \beta| \leq m+1, \, |\alpha| \leq m, \, |\beta| \leq m \right\}$$

*is the admissible set of 3-tuples $(j,\alpha,\beta)$ with $\alpha,\beta$ being multi-indices for the orders of space-time derivatives. In particular, the constraint $|\alpha| + j > 0$ excludes such terms as $\left\|(B_0)(\partial^\beta V)\right\|_{L^2}$ from the sum.*

REMARK 4.3. *In two dimensions, the compressible Euler equations naturally satisfy the structural assumption regarding the vorticity equation (4.8)—consult (2.7). In three dimensions, the vorticity equation (2.8) is a lower order perturbation of (4.8) and our argument remains essentially valid.*

*Similarly, one can manipulate the two-dimensional (2D) RSW equations to obtain*

$$\partial_t(\rho - \nabla\times u) + \nabla\cdot(u(\rho - \nabla\times u)) = 0.$$

*Due to change of variables $\varepsilon r = p(1 + \varepsilon\rho)$, this is equivalent to*

$$\partial_t(r - \nabla\times u) + \nabla\cdot(u(r - \nabla\times u)) + \left[p^{-1}(\varepsilon r)p'(p^{-1}(\varepsilon r)) - 1\right]\frac{1}{\varepsilon}\nabla\cdot u = 0.$$

*Upon rescaling (4.2), it leads to*

$$\partial_t(\check{r} - \nabla\times\check{u}) + \nabla\cdot(\check{u}(\check{r} - \nabla\times\check{u})) + \left[p^{-1}(\check{r})p'(p^{-1}(\check{r})) - 1\right]\nabla\cdot\check{u} = 0,$$

*which is again a lower order perturbation of (4.8).*

*Therefore, for simplicity, we focus only on a "vorticity equation" in the form of* (4.8) *while omitting any extra terms containing lower order derivatives whose* $H^{m-1}$ *norms vanish at order* $O(\|V\|_{H^m}^2)$ *around* $V \equiv 0$. *Such a simplification leads only to nonessential changes in the energy estimate* (4.10).

*Proof.* We establish the following estimates:

- $H^{m-1}$ norm of vorticity $\mathcal{K}[V]$, i.e.,

$$(4.11) \qquad\qquad \|\mathcal{K}[V]\|_{H^{m-1}} \lesssim \mathsf{S}.$$

For simplicity, define

$$\omega := \mathcal{K}[V].$$

Take spatial derivative $\partial_x^\alpha$ with $|\alpha| \leq m-1$ on (4.8),

$$\partial_t \partial_x^\alpha \omega + \sum_{j=1}^{\mathbf{D}} \tilde{B}_j \partial_{x_j} \partial_x^\alpha \omega + \sum_{j=1}^{\mathbf{D}} \left( \partial_x^\alpha \partial_{x_j} (\tilde{B}_j \omega) - \tilde{B}_j \partial_x^\alpha \partial_{x_j} \omega \right) = 0.$$

Then, take the $L_x^2$-inner product with $\partial_x^\alpha \omega$,

$$(4.12)$$
$$0 = \frac{1}{2} \partial_t \|\partial_x^\alpha \omega\|_{L_x^2}^2 + \left\langle \partial_x^\alpha \omega, \sum_{j=1}^{\mathbf{D}} \tilde{B}_j \partial_{x_j} \partial_x^\alpha \omega \right\rangle_{L_x^2} + \left\langle \partial_x^\alpha \omega, \sum_{j=1}^{\mathbf{D}} \left( \partial_x^\alpha \partial_{x_j} (\tilde{B}_j \omega) - \tilde{B}_j \partial_x^\alpha \partial_{x_j} \omega \right) \right\rangle_{L_x^2}$$

$$=: \frac{1}{2} \partial_t \|\partial_x^\alpha \omega\|_{L_x^2}^2 + I_1 + I_2.$$

Apply the Stokes theorem on the $I_1$ term above, using the assumptions that all $B_j$ are symmetric matrices and $\tilde{B}_{\mathsf{n}}\big|_{\partial\Omega} \geq 0$,

$$I_1 \geq -\frac{1}{2} \left\langle \partial_x^\alpha \omega, \left( \sum_{j=1}^{\mathbf{D}} \partial_{x_j} \tilde{B}_j \right) \partial_x^\alpha \omega \right\rangle_{L_x^2}.$$

Apply the Hölder inequality to the RHS above and the $II$ term in (4.12),

$$-I_1 - I_2 \lesssim \|\partial_x^\alpha \omega\|_{L_x^2} \sum_{(j,\alpha,\beta) \in \mathbf{A}_m} \left\| (\partial_{t,x}^\alpha \tilde{B}_j)(\partial_{t,x}^\beta V) \right\|_{L_x^2}.$$

Therefore, upon integrating (4.12) in time and summing over all $|\alpha| \leq m-1$, we obtain (4.11).

- $L^2$ norms of time derivatives $\partial_t^n V$ $(n = 0, 1, \dots, m)$, i.e.,

$$(4.13) \qquad\qquad \|\partial_t^n V\|_{L^2} \lesssim S, \qquad n = 0, 1, \dots, m.$$

For simplicity, define

$$V_n := \partial_t^n V.$$

Take time derivative $\partial_t^n$ on (4.5),

$$B_0 \partial_t V_n + \sum_{j=1}^{\mathbf{D}} B_j \partial_{x_j} V_n$$

$$+ \sum_{j=0}^{\mathbf{D}} \left( \partial_t^n (B_j \partial_{x_j} V) - B_j \partial_t^n \partial_{x_j} V \right) + \mathcal{L}[V_n] = 0.$$

Then, take the $L_x^2$-inner product with $V_n$,

(4.14)

$$0 = \frac{1}{2}\partial_t\|V_n\|_{L_{x,B_0}^2}^2 - \frac{1}{2}\Big\langle V_n, \partial_t B_0 V_n\Big\rangle_{L_x^2} + \Big\langle V_n, \sum_{j=1}^{\mathbf{D}} B_j\partial_{x_j}V_n\Big\rangle_{L_x^2}$$

$$+ \Big\langle V_n, \sum_{j=0}^{\mathbf{D}}\big(\partial_t^n(B_j\partial_{x_j}V) - B_j\partial_t^n\partial_{x_j}V\big)\Big\rangle_{L_x^2} + \langle V_n, \mathcal{L}[V_n]\rangle_{L_x^2}$$

$$=: \frac{1}{2}\partial_t\|V_n\|_{L_{x,B_0}^2}^2 + I_3 + I_4 + I_5 + \langle V_n, \mathcal{L}[V_n]\rangle_{L_x^2}.$$

Here, $\|\cdot\|_{L_{x,B_0}^2}$ indicates a $B_0$-weighted $L_x^2$ norm.

Since the boundary condition is time-independent, we obtain that $V_n$ satisfies the same boundary condition (4.6) and in particular, by (4.7), makes the last term above nonnegative:

$$\langle V_n, \mathcal{L}[V_n]\rangle_{L_x^2} \geq 0.$$

The $I_3, I_4, I_5$ terms are estimated in the same fashion as that of estimating $I_1, I_2$ in (4.12). In particular, the zeroth derivative of $B_0$ does not appear in the estimate.

Therefore, upon integrating (4.14) in time and using the fact that all eigenvalues of $B_0$ are located on $[1/2, 2]$, we obtain (4.13).

We remark that, for the highest time derivative $\partial_t^m V$, it is only in $L^2(\Omega)$, and thus more care is needed to perform the above estimates: first perform the energy method to get an upper bound for

$$\partial_t\big\|\partial_t^{m-1}V(t+\tau,\cdot) - \partial_t^{m-1}V(t,\cdot)\big\|_{L_x^2}^2$$

with just enough regularity and, in particular, with the boundary integral well defined; then integrate in time to get an estimate for $\big\|\partial_t^{m-1}V(t+\tau,\cdot) - \partial_t^{m-1}V(t,\cdot)\big\|_{L_x^2}$; and finally divide it by $\tau$ and let $\tau \to 0$ to obtain an estimate on $\|\partial_t^m V\|_{L_x^2}$.

- Finally, the solution's $H_x^m$ norm

$$\|V\|_{H_x^m} \lesssim \mathsf{S}.$$

We will repeatedly use elliptic estimates (2.11) for $U = V, \partial_t V, \partial_t^2 V, \ldots, \partial_t^{m-1}$. We will also repeatedly use estimate (4.13) and estimate

(4.15) $$\|\mathcal{K}[\partial_t^n V]\|_{H_x^{m-1-n}} \lesssim \mathsf{S}, \qquad n = 0, 1, \ldots, m-1,$$

as a direct consequence of (4.11) and (4.8).

$$\begin{aligned}\|V\|_{H_x^m} &\lesssim \|\mathcal{L}[V]\|_{H_x^{m-1}} + \|\mathcal{K}[V]\|_{H_x^{m-1}} + \|V\|_{L_x^2} && \text{by (2.11)}\\ &\lesssim \|\mathcal{L}[V]\|_{H_x^{m-1}} + \mathsf{S} && \text{by (4.13), (4.15)}\\ &\lesssim \|\partial_t V\|_{H_x^{m-1}} + \mathsf{S},\end{aligned}$$

where the last inequality is due to (4.5) and $B_0 \in [1/2, 2]$ with the nonlinear terms absorbed into $\mathsf{S}$; continuing,

$$\begin{aligned}\ldots &\lesssim \|\mathcal{L}[\partial_t V]\|_{H_x^{m-2}} + \|\mathcal{K}[\partial_t V]\|_{H_x^{m-2}} + \|\partial_t V\|_{L_x^2} + \mathsf{S} && \text{by (2.11)}\\ &\lesssim \|\mathcal{L}[\partial_t V]\|_{H_x^{m-2}} + \mathsf{S} && \text{by (4.13), (4.15)}\\ &\lesssim \|\partial_t^2 V\|_{H_x^{m-2}} + \mathsf{S},\end{aligned}$$

where the last inequality is due to (4.5) and $B_0 \in [1/2, 2]$ with the nonlinear terms absorbed into $\mathsf{S}$; continuing,

$$\ldots \lesssim \|\partial_t^3 V\|_{H_x^{m-3}} + \mathsf{S}$$
$$\ldots \lesssim \|\partial_t^4 V\|_{H_x^{m-4}} + \mathsf{S}$$
$$\ldots\ldots\ldots \qquad\qquad\qquad\qquad\qquad\qquad \text{inductively}$$
$$\ldots \lesssim \|\partial_t^m V\|_{H_x^0} + \mathsf{S}$$
$$\lesssim \mathsf{S} \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{by (4.13).} \qquad \square$$

**5. Extension to the RSW equations.** We extend the above framework to the 2D RSW equations in a very natural way. Recall the RSW equations (1.2) with $F = -u^\perp$ and constraints (1.3), i.e.,

$$\partial_t \rho + \nabla \cdot (\rho u) + \frac{1}{\varepsilon} \nabla \cdot u = 0, \tag{5.1}$$

$$\partial_t u + u \cdot \nabla u + \frac{1}{\varepsilon} \nabla \rho + \frac{1}{\varepsilon} u^\perp = 0, \tag{5.2}$$

$$u \cdot \mathsf{n}\big|_{\partial\Omega} = 0, \qquad \int_\Omega \rho = 0. \tag{5.3}$$

The analogue between the above system and the compressible Euler equations is as follows. The pressure law for the RSW equations is $p(\rho) = \frac{1}{2}\rho^2$, which is due to the gravitational force. The singular parameter $\varepsilon$ is the 2D Froude number and plays the same role as the Mach number. The Rossby is also set to scale at the same order of $\varepsilon$.

The elliptic operators differ from their counterparts (2.4), (2.5) by lower order perturbations,

$$\widetilde{\mathcal{L}}[U] := \begin{pmatrix} \nabla \cdot u \\ \nabla \rho + u^\perp \end{pmatrix},$$

$$\widetilde{\mathcal{K}}[U] := \rho - (\partial_x u_2 - \partial_y u_1) = \rho - \nabla^\perp \cdot u,$$

so that an analogue of (2.6) still holds true:

$$\widetilde{\mathcal{K}}\widetilde{\mathcal{L}} = 0.$$

The RSW system is therefore endowed with a $\varepsilon$-free vorticity equation, similar to (2.7),

$$\partial_t \widetilde{\mathcal{K}}[U] + \nabla \cdot (u \, \widetilde{\mathcal{K}}[U]) = 0. \tag{5.4}$$

We can rewrite the RSW system in a more compact form,

$$\partial_t U + \widetilde{\mathcal{N}}(U, \nabla U) = -\frac{1}{\varepsilon} \widetilde{\mathcal{L}}[U], \quad u \cdot \mathsf{n}\big|_{\partial\Omega} = 0, \quad \int_\Omega \rho = 0, \tag{5.5}$$

where the nonlinear term

$$\widetilde{\mathcal{N}}(U_1, \nabla U_2) := \begin{pmatrix} u_1 \cdot \nabla \rho_2 + \rho_1 \nabla \cdot u_2 \\ u_1 \cdot \nabla u_2 \end{pmatrix}. \tag{5.6}$$

**5.1. Analogue to section 2: Elliptic estimates and projections.** The analogue of elliptic estimates (2.11) is quite straightforward since $\widetilde{\mathcal{L}}$, $\widetilde{\mathcal{K}}$ defined here are only lower order perturbations of their counterparts in section 2:

$$(5.7) \quad \|U\|_{H^m(\Omega)} \le C \left( \|\widetilde{\mathcal{L}}[U]\|_{H^{m-1}(\Omega)} + \|\widetilde{\mathcal{K}}[U]\|_{H^{m-1}(\Omega)} + \|u\|_{L^2(\Omega)} \right) \quad \text{for } U \in \mathbf{X}^m$$

with the solution space $\mathbf{X}^m$ defined in section 2.

For the projection operators, we essentially need to find an elliptic PDE to define $\widetilde{\mathcal{P}}$, $\widetilde{\mathcal{Q}}$ which is as convenient as (2.14). Indeed, since $(\rho, u) \in \mathrm{Ker}\widetilde{\mathcal{L}}$ iff $u = (\nabla\rho)^{\perp}$, we characterize $\widetilde{\mathcal{P}}$ in the following way: for $U \in H^m$ (not necessarily in $\mathbf{X}^m$),

$$\widetilde{\mathcal{P}}[U] := \begin{pmatrix} \phi \\ \nabla^{\perp}\phi \end{pmatrix} \quad \text{with} \quad \phi := (1-\Delta)^{-1}_{\mathrm{QG}}(\rho - \nabla^{\perp} \cdot u).$$

Here and below, subscript QG indicates constraints of constant boundary value and zero mean associated with $(1-\Delta)^{-1}$. Indeed, the definition of $(1-\Delta)^{-1}_{\mathrm{QG}}$ is based on the following lemma.

LEMMA 5.1. *For any $f \in H^n(\Omega)$, there exists a unique solution $w$, denoted by $w = (1-\Delta)^{-1}_{\mathrm{QG}}[f]$, to the elliptic PDE,*

$$(1-\Delta)w = f,$$

$$w\Big|_{\partial\Omega} = \text{constant}, \quad \int_{\Omega} w = 0,$$

*with $w$ satisfying estimate*

$$\|w\|_{H^{n+2}(\Omega)} \le C\|f\|_{H^n(\Omega)}.$$

*Proof.* By standard elliptic PDE theory, the following equation with Dirichlet boundary condition

$$(1-\Delta)w_1 = f,$$

$$w_1\Big|_{\partial\Omega} = 0$$

admits a unique solution $w_1$ with estimates

$$\|w_1\|_{H^{n+2}(\Omega)} \le C\|f\|_{H^n(\Omega)}.$$

Thus, define

$$w := w_1 - \frac{1}{|\Omega|}\int_{\Omega} w_1$$

to be the solution to the original PDE. It is obvious that the $H^{n+2}$ estimate of $w$ follows that of $w_1$ and the Poincaré inequalities. This estimate also guarantees the uniqueness of $w$. $\quad\square$

In a more compact and essential form, the projections are given as

$$(5.8) \qquad\qquad \widetilde{\mathcal{P}} := \widetilde{\mathcal{K}}^*(\widetilde{\mathcal{K}}\widetilde{\mathcal{K}}^*)^{-1}_{\mathrm{QG}}\widetilde{\mathcal{K}}, \qquad \widetilde{\mathcal{Q}} := \mathcal{I} - \widetilde{\mathcal{P}},$$

where $\widetilde{\mathcal{K}}^*$ is the formal adjoint of $\widetilde{\mathcal{K}}$ so that for scalar-valued function $\phi$

$$\widetilde{\mathcal{K}}^*[\phi] = \begin{pmatrix} \phi \\ (\nabla \phi)^\perp \end{pmatrix} \quad \text{and}$$

$$\widetilde{\mathcal{K}}\widetilde{\mathcal{K}}^*[\phi] = (1 - \Delta)\phi.$$

The identities (5.8) suggest a parallel argument in linear algebra and, in particular, the method of least squares. Indeed, by definition, we have

$$(5.9) \qquad \begin{cases} \widetilde{\mathcal{L}}\widetilde{\mathcal{P}} = \widetilde{\mathcal{P}}\widetilde{\mathcal{L}} = 0, & \widetilde{\mathcal{L}}\widetilde{\mathcal{Q}} = \widetilde{\mathcal{Q}}\widetilde{\mathcal{L}} = \widetilde{\mathcal{L}}, \\ \widetilde{\mathcal{K}}\widetilde{\mathcal{P}} = \widetilde{\mathcal{K}}, & \widetilde{\mathcal{P}}\widetilde{\mathcal{K}}^* = \widetilde{\mathcal{K}}^*, \quad \widetilde{\mathcal{K}}\widetilde{\mathcal{Q}} = 0, \quad \widetilde{\mathcal{Q}}\widetilde{\mathcal{K}}^* = 0. \end{cases} \qquad (\text{note: } \widetilde{\mathcal{L}}^* = -\widetilde{\mathcal{L}})$$

As to the boundedness of $\widetilde{\mathcal{P}}$ and $\widetilde{\mathcal{Q}}$, let us point out a technique detail: due to the rather unusual constraints put on $(1 - \Delta)_{\mathrm{QG}}^{-1}$, we cannot prove a clean version of the Pythagorean theorem $\|U\|_{L^2} = \|\widetilde{\mathcal{P}}[U]\|_{L^2} + \|\widetilde{\mathcal{Q}}[U]\|_{L^2}$; however, for the mere sake of elliptic estimates, it suffices to use Lemma 5.1 to deduce the following analogue of (2.17):

$$(5.10) \qquad \|\widetilde{\mathcal{P}}[U]\|_{H^m} \le C\|U\|_{H^m} \quad \text{for } U \in H^m.$$

Thanks to duality, applying (5.7), (5.9) on $\widetilde{\mathcal{Q}}[U]$, we have the analogue of (2.18),

$$(5.11) \qquad \|\widetilde{\mathcal{Q}}[U]\|_{H^m} \le C(\|\widetilde{\mathcal{L}}[U]\|_{H^{m-1}} + \|\widetilde{\mathcal{Q}}[U]\|_{L^2}) \le C'\|U\|_{H^m} \quad \text{for } U \in \mathbf{X}^m.$$

But we will need the following lemma as a stronger version of (5.11) and an analogue of Lemma 2.4.

LEMMA 5.2. *For any $U \in \mathbf{X}^m$ ($m \ge 1$),*

$$\|\widetilde{\mathcal{Q}}[U]\|_{H^m} \le C\|\widetilde{\mathcal{L}}[U]\|_{H^{m-1}}.$$

*Proof.* The first inequality of (5.11) being established, it suffices to show that

$$\|\widetilde{\mathcal{Q}}[U]\|_{L^2} \le C\|\widetilde{\mathcal{L}}[U]\|_{H^{m-1}} \quad \text{for } U \in \mathbf{X}^m.$$

Suppose not. Then, there exists a sequence of functions $\{U_N\}_{N=1}^\infty$ in $\mathbf{X}^m$ so that

$$\|\widetilde{\mathcal{Q}}[U_N]\|_{L^2} \ge N\|\widetilde{\mathcal{L}}[U_N]\|_{H^{m-1}}, \quad N = 1, 2, 3, \ldots.$$

Upon rescaling, we can choose to have

$$\|\widetilde{\mathcal{Q}}[U_N]\|_{L^2} = 1, \quad \|\widetilde{\mathcal{L}}[U_N]\|_{H^{m-1}} \le \frac{1}{N}, \quad N = 1, 2, 3, \ldots.$$

Combined with (5.11), this implies

$$\{\widetilde{\mathcal{Q}}[U_N]\}_{N=1}^\infty \text{ is a bounded sequence in } H^m,$$

and by weak compactness of $H^m$ and compact imbedding of $H^m$ into $L^2$, there exists a $U_\infty \in H^m$ (actually also in $\mathbf{X}^m$) that is the weak limit of $\{\widetilde{\mathcal{Q}}[U_N]\}$ in $H^m$ and the strong limit in $L^2$. Then, respectively,

$$\text{weak limit in } H^m \implies \begin{cases} \|\widetilde{\mathcal{L}}[U_\infty]\|_{H^{m-1}} \le \liminf_{N\to\infty} \|\widetilde{\mathcal{L}}\widetilde{\mathcal{Q}}[U_N]\|_{H^{m-1}} = \liminf_{N\to\infty} \frac{1}{N} = 0, \\ \|\widetilde{\mathcal{K}}[U_\infty]\|_{H^{m-1}} \le \liminf_{N\to\infty} \|\widetilde{\mathcal{K}}\widetilde{\mathcal{Q}}[U_N]\|_{H^{m-1}} = 0, \end{cases}$$

$$\text{strong limit in } L^2 \implies \|U_\infty\|_{L^2} = \lim_{N\to\infty} \|\widetilde{\mathcal{Q}}[U_N]\|_{L^2} = 1.$$

In other words,

(5.12) $$U_\infty \in \text{Ker}\widetilde{\mathcal{K}} \cap \text{Ker}\widetilde{\mathcal{L}} \cap \mathbf{X}^m \quad \text{and} \quad \|U_\infty\|_{L^2} = 1.$$

This would be a contradiction. In fact, let $U_\infty = \left(\begin{smallmatrix} \rho_\infty \\ u_\infty \end{smallmatrix}\right)$. Then,

$$U_\infty \in \text{Ker}\widetilde{\mathcal{K}} \implies \rho_\infty = \nabla^\perp \cdot u_\infty,$$
$$U_\infty \in \text{Ker}\widetilde{\mathcal{L}} \implies \nabla\rho_\infty + u_\infty^\perp = 0,$$

which imply the Helmholtz equation $(1 - \Delta)\rho_\infty = 0$. Meanwhile, $U_\infty \in \mathbf{X}^m$ implies $\int_\Omega \rho_\infty = 0$ and, together with the second equation above, implies $\nabla^\perp\rho_\infty \cdot \mathsf{n}\big|_{\partial\Omega} = 0 \implies \rho_\infty\big|_{\partial\Omega} = \text{constant}$. In other words, $\rho_\infty$ is in the image of $(1 - \Delta)_{\text{QG}}^{-1}$,

$$\rho_\infty = (1 - \Delta)_{\text{QG}}^{-1}0 = 0.$$

Then, we have $U_\infty = 0$, which contradicts $\|U\|_{L^2} = 1$ in (5.12)!     □

Combine this lemma with Lemma 2.6, and easily prove the following analogue of Theorem 2.5.

THEOREM 5.3. *Consider the RSW equations (5.1)–(5.3) under the same assumptions as in Theorem 1.3. Then, (1.7) holds true, that is,*

$$\left\| \frac{1}{t} \int_0^t U^Q d\tau \right\|_{H^m(\Omega)} \le C\varepsilon \left(1 + \frac{1}{t}\right) \qquad for\ t \in (0, T].$$

*Here, $C$ depends only on $\max_{t\in[0,T]} \|U\|_{H^m(\Omega)}$ and $m$, and it is independent of $\varepsilon$ and $t$ otherwise.*

**5.2. Analogue to section 3: Estimates on nonlinear interaction and strong convergence.** The goal of this subsection is to prove the following analogue of Theorem 3.1.

THEOREM 5.4. *Consider the 2D RSW equations (5.5). Assume a solution exists classically: $U(t, x) \in \cap_{j=0}^m \mathbf{C}^j([0, T]; H^{m-j}(\Omega))$ with $m > 5$. Also assume a nonvacuum condition $\|\varepsilon\rho\|_{\mathbf{C}([0,T];H^m)} \le 1/2$. Then, with projection $\widetilde{\mathcal{P}}$ defined in (5.8), there exists a quasi-geostrophic flow $\overline{U}$ so that*

$$\max_{t\in[0,T]} \|\widetilde{\mathcal{P}}[U] - \overline{U}\|_{m-3} \le \varepsilon C.$$

*Here, $C$ depends only on $\max_{t\in[0,T]} \|U\|_{H^m(\Omega)}$, $m$, and $T$, and it is independent of $\varepsilon$.*

*In particular, $\overline{U} = \left(\begin{smallmatrix} \overline{\rho} \\ \overline{u} \end{smallmatrix}\right)$ is uniquely determined from*

(5.13) $$\overline{\rho} = (1 - \Delta)_{\text{QG}}^{-1}\theta, \qquad \overline{u} = \nabla^\perp(1 - \Delta)_{\text{QG}}^{-1}\theta,$$

*with $\theta$ solving the quasi-geostrophic equations*

(5.14) $$\partial_t\theta + \overline{u}\cdot\nabla\theta = 0,$$

(5.15) $$\theta_0 = \rho_0 - \nabla^\perp \cdot u_0.$$

To prepare for the proof of this theorem, we apply $\widetilde{\mathcal{P}}$ on (5.5). By duality relations (5.9) and the bilinearity of $\widetilde{\mathcal{N}}$, we have

(5.16) $$\begin{aligned} &\partial_t U^P + \widetilde{\mathcal{P}}\widetilde{\mathcal{N}}(U^P, \nabla U^P) + \widetilde{\mathcal{P}}\widetilde{\mathcal{N}}(U^Q, \nabla U^Q) \\ &\quad + \widetilde{\mathcal{P}}\widetilde{\mathcal{N}}(U^Q, \nabla U^P) + \widetilde{\mathcal{P}}\widetilde{\mathcal{N}}(U^P, \nabla U^Q) = 0. \end{aligned}$$

Thus, there are two types of nonlinear interactions to be studied,

$$\text{``fast-fast''} \qquad \widetilde{\mathcal{P}}\widetilde{\mathcal{N}}(U^Q, U^Q),$$

$$\text{``fast-slow''} \qquad \widetilde{\mathcal{P}}\widetilde{\mathcal{N}}(U^Q, U^P) + \mathcal{P}\mathcal{N}(U^P, U^Q).$$

Lemma 3.2 regarding cancellation of the "fast-fast" interaction under the projection $\mathcal{P}$ is still valid for RSW equations due to a simple observation: the vorticity equation (5.4) results from applying $\widetilde{\mathcal{K}}$ to (5.5) and using $\widetilde{\mathcal{K}}\widetilde{\mathcal{L}} = 0$. In other words, the identity

(5.17) $$\widetilde{\mathcal{K}}\widetilde{\mathcal{N}}(U, \nabla U) = \nabla\cdot(u\,\widetilde{\mathcal{K}}[U]) \qquad \text{with } \widetilde{\mathcal{N}} \text{ defined in (5.6)}$$

is true regardless of the equation. Now, let $U = U^Q$ be the fast part of a solution. Then, by the duality relations (5.9), $\widetilde{\mathcal{K}}[U^Q] = 0$. Thus, the above identity implies

$$\widetilde{\mathcal{K}}\widetilde{\mathcal{N}}(U^Q, \nabla U^Q) = \nabla\cdot(u^Q\,\widetilde{\mathcal{K}}[U^Q]) \equiv 0.$$

Together with (5.8), it leads to the following lemma,

LEMMA 5.5. *For any $U^Q \in \mathrm{Img}\widetilde{\mathcal{Q}}$ with sufficient regularity,*

$$\widetilde{\mathcal{P}}\widetilde{\mathcal{N}}(U^Q, \nabla U^Q) \equiv 0.$$

The following lemma regarding the $O(\varepsilon)$ estimate of the "fast-slow" interaction upon time averaging follows exactly the same proof as its counterpart for Euler equations, Lemma 3.3. We skip its proof, highlighting only that the key estimate necessary for the proof

$$\|W(t, \cdot)\|_{H^m(\Omega)} \leq \varepsilon C(t+1),$$

$$\text{where} \quad W(t, \cdot) := \int_0^t U^Q(\tau, \cdot)d\tau,$$

is still valid thanks to Theorem 5.3.

LEMMA 5.6. *For any $U = U^Q + U^P \in \mathbf{X}^m$ with $m > \mathbf{D}/2 + 3$ that solves (5.5),*

$$\left\|\int_0^t \widetilde{\mathcal{P}}\widetilde{\mathcal{N}}(U^P, \nabla U^Q) + \widetilde{\mathcal{P}}\widetilde{\mathcal{N}}(U^Q, \nabla U^P)\, d\tau\right\|_{H^{m-2}} \leq \varepsilon C(t+1)^2.$$

*Here, $C$ depends only on $\max_{t\in[0,T]} \|U\|_{H^m(\Omega)}$ and $m$, and it is independent of $\varepsilon$ and $t$ otherwise.*

Now we are ready to prove Theorem 5.4.

*Proof of Theorem* 5.4. In this proof, we assume the existence time of classical solutions to the RSW and quasi-geostrophic equations depends solely on the size of initial data and is otherwise independent of $\varepsilon$. For a detailed proof on uniform energy estimates, consult section 4.

Apply Lemmas 5.5 and 5.6 on the time integral of (5.16) to obtain,

$$\text{for} \ -\xi_1(t, \cdot) := \int_0^t \partial_t U^P + \widetilde{\mathcal{P}}\widetilde{\mathcal{N}}(U^P, \nabla U^P)\, d\tau,$$

$$\|\xi_1\|_{H^{m-2}(\Omega)} \leq \varepsilon C(t+1)^2.$$

Apply $\widetilde{\mathcal{K}}$ on the time derivative of the above equation and employ (5.9), (5.17), and $\nabla \cdot u^P = 0$ to obtain

$$(5.18) \qquad -\partial_t \xi = \partial_t \widetilde{\mathcal{K}}[U^P] + u^P \cdot \nabla \widetilde{\mathcal{K}}[U^P],$$

$$(5.19) \qquad \text{where } \xi := \widetilde{\mathcal{K}}[\xi_1] \quad \text{so that} \quad \|\xi\|_{H^{m-3}(\Omega)} \leq C\varepsilon.$$

Now, define

$$\delta := \theta - \widetilde{\mathcal{K}}[U^P].$$

Note that, by $\overline{U} = \nabla^\perp (1 - \Delta)_{\text{QG}}^{-1}[\theta]$ and $U^P = \nabla^\perp (1 - \Delta)_{\text{QG}}^{-1} \widetilde{\mathcal{K}}[U^P]$, we have

$$\overline{U} - U^P = \nabla^\perp (1 - \Delta)_{\text{QG}}^{-1}[\delta],$$

and thus, by Lemma 5.1,

$$(5.20) \qquad \|\overline{U} - U^P\|_{H^{m-3}} \leq C\|\delta\|_{H^{m-4}}.$$

We subtract (5.18) from (5.14) to obtain an equation for $\delta$,

$$\partial_t \xi = \partial_t \delta + u^P \cdot \nabla \delta + (\overline{u} - u^P) \cdot \nabla \theta.$$

Then, take spatial derivative $\partial^\alpha$ with $|\alpha| \leq m - 4$ on the above equation and obtain

$$(5.21) \qquad \partial_t \xi^\alpha = \partial_t \delta^\alpha + u^P \cdot \nabla \delta^\alpha + f^\alpha,$$
$$\text{where} \quad \xi^\alpha := \partial^\alpha \xi, \ \delta^\alpha := \partial^\alpha \delta$$
$$\text{and} \quad f^\alpha := \partial^\alpha (u^P \cdot \nabla \delta) - u^P \cdot \nabla \delta^\alpha + \partial^\alpha \big( (\overline{u} - u^P) \cdot \nabla \theta \big).$$

Since $|\alpha| \leq m - 4$, by the calculus inequalities (1.12) and estimate (5.20), we obtain

$$(5.22) \qquad \|f^\alpha\|_{L^2} \leq C\|\delta\|_{H^{m-4}}.$$

Since $\partial_t \xi$ is an $O(1)$ term, we further rewrite (5.21) as

$$(5.23) \qquad 0 = \partial_t (\delta^\alpha - \xi^\alpha) + u^P \cdot \nabla (\delta^\alpha - \xi^\alpha) + u^P \cdot \nabla \xi^\alpha + f^\alpha,$$

where, by (5.19) and $|\alpha| \leq m - 4$, it holds true that

$$(5.24) \qquad \|u^P \cdot \nabla \xi^\alpha\|_{L^2} \leq C\varepsilon.$$

Now, we are ready to perform energy estimates. Take the $L^2(\Omega)$-inner product of (5.23) with $\delta^\alpha - \xi^\alpha$,

$$0 = \frac{1}{2} \partial_t \Big\langle \delta^\alpha - \xi^\alpha, \, \delta^\alpha - \xi^\alpha \Big\rangle + \Big\langle u^P \cdot \nabla (\delta^\alpha - \xi^\alpha), \, (\delta^\alpha - \xi^\alpha) \Big\rangle$$
$$+ \Big\langle u^P \cdot \nabla \xi^\alpha + f^\alpha, \, \delta^\alpha - \xi^\alpha \Big\rangle.$$

Since $u^P$ is div-free, we apply the Stokes theorem to the second term on the RHS,

$$\Big\langle u^P \cdot \nabla (\delta^\alpha - \xi^\alpha), \, (\delta^\alpha - \xi^\alpha) \Big\rangle_{L^2(\Omega)} = \frac{1}{2} \int_{\partial\Omega} (u^P \cdot \mathsf{n}) |\delta^\alpha - \xi^\alpha|^2 = 0,$$

where the last equality is due to the solid-wall boundary condition.

Combining the above two equalities with estimates (5.22), (5.24), we arrive at

$$\frac{1}{2}\partial_t\langle\delta^\alpha - \xi^\alpha, \delta^\alpha - \xi^\alpha\rangle \leq C\|\delta - \xi^\alpha\|_{H^{m-4}}(\|\delta - \xi^\alpha\|_{H^{m-4}} + \varepsilon).$$

Sum all such inequalities over all $\alpha$ with $|\alpha| \leq m - 4$ to obtain

$$\partial_t\|\delta - \xi\|_{H^{m-4}}^2 \leq C\|\delta - \xi\|_{H^{m-4}}(\|\delta - \xi\|_{H^{m-4}} + \varepsilon).$$

Finally, apply the Gronwall inequality together with the initial conditions $\delta(0, \cdot) = \xi(0, \cdot) \equiv 0$ to arrive at

$$\max_{t\in[0,T]}\|\delta - \xi\|_{m-4} \leq C\varepsilon.$$

Together with estimates (5.19), (5.20), this proves the theorem.  $\square$

**6. Effects on transport of passive scalars.** We now prove Corollary 1.4. Let's recall that we aim to estimate the difference of the solution to

$$\partial_t s + u\cdot\nabla s = 0 \tag{6.1}$$

from the solution to

$$\partial_t\overline{s} + \overline{u}\cdot\nabla\overline{s} = 0, \tag{6.2}$$

both subject to the same initial data $\overline{s}_0(\cdot) = s_0(\cdot)$.

Subtract these two equations, and rewrite in terms of $\delta := s - \overline{s}$ as

$$\partial_t\delta + \overline{u}\cdot\nabla\delta + (u - \overline{u})\cdot\nabla s = 0. \tag{6.3}$$

Then, following the same time-integral technique, we define

$$w(t, \cdot) := \int_0^t (u - \overline{u})\,d\tau \qquad \text{and}$$

$$\xi(t, \cdot) := \int_0^t (u - \overline{u})\cdot\nabla s\,d\tau.$$

Upon integrating by parts,

$$\xi(t, \cdot) = w\cdot\nabla s\Big|_0^t - \int_0^t w\cdot\nabla(\partial_t s)\,d\tau. \tag{6.4}$$

To estimate $w$ terms on the RHS, we split it using $u = u^Q + u^P$:

$$w(t, \cdot) = \int_0^t u^Q\,d\tau + \int_0^t (u^P - \overline{u})\,d\tau.$$

Apply (1.6) (resp., (1.7)) on the first term, and apply (1.4) (resp., (1.8)) on the second term. Then, we show that, on some finite time interval $[0, T]$,

$$\|w\|_{H^m} \leq C\varepsilon.$$

For $s$ terms in (6.4), perform straightforward estimation using (6.1). Thus, locally in time,

$$\|\partial_t\nabla s\|_{H^{m-2}} \leq C, \quad \|\nabla s\|_{H^{m-1}} \leq C.$$

Plug the above two estimates into (6.4) to obtain, on some finite time interval $[0, T]$,

$$(6.5) \qquad \|\xi\|_{H^{m-2}} \leq C\varepsilon.$$

Now, use $\xi$ to rewrite (6.3) as

$$\partial_t \delta + \overline{u} \cdot \nabla \delta + \partial_t \xi = 0$$

and further

$$\partial_t (\delta + \xi) + \overline{u} \cdot \nabla (\delta + \xi) - \overline{u} \cdot \nabla \xi = 0.$$

Then, we can perform energy estimates on the above equation for $(\delta + \xi)$ up to $(m-3)$rd order spatial derivatives. Note that the boundary condition $\overline{u} \cdot \mathsf{n}\big|_{\partial\Omega} = 0$ will make the boundary integral vanish; and also note that, by (6.5), the $-\overline{u} \cdot \nabla \xi$ term is bounded by $C\varepsilon$ in the $H^{m-3}$ norm. So, a routine work of energy method together with $\delta\big|_{t=0} = \xi\big|_{t=0} \equiv 0$ will imply, on some finite time interval $[0, T]$,

$$\|\delta + \xi\|_{H^{m-3}} \leq C\varepsilon.$$

Together with (6.5), this leads to the conclusion of Corollary 1.4.

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