# 1. Introduction

The study of orbits of symplectic maps under iteration has a rich history motivated by basic questions in symplectic geometry and their importance in applications – celestial mechanics, plasma physics, accelerator dynamics, condensed matter physics and fluid flow, for example. The present work is a research monograph that considers particular interesting questions about symplectic maps on $\mathbb{R}^{2n}$ with $n \geq 1$. The monograph consists of three parts: a general theory for bifurcating period-$q$ points of equivariant symplectic maps, introduction of a singularity theory framework for equivariant gradient bifurcation problems which is used to classify singularities of bifurcating period-$q$ points of symplectic maps and thirdly a compendium – much of which is contained in a sequence of appendices – of basic questions and results for symplectic maps on $\mathbb{R}^{2n}$ and their generating functions.

Symmetry arises in two ways in the analysis of period-$q$ points. A period-$q$ orbit of a symplectic map has a natural cyclic ($\mathbb{Z}_q$) symmetry; that is, an orbit can start at any of the $q$ distinct points that make up the period-$q$ orbit. Period-$q$ points can therefore be characterized as fixed points of a $\mathbb{Z}_q$-equivariant map on $\mathbb{R}^{nq}$. When the symplectic map is itself equivariant we call this a spatial symmetry (that is, it is independent of discrete time). A general framework to account for the two types of symmetry is introduced in Chapters 2 and 8 respectively.

When the map depends on parameters the periodic points will undergo bifurcations as the parameters are varied. To classify the bifurcations we introduce a singularity theory framework in Chapter 3 with special cases treated in Chapters 4 and 6. These results are self-contained and have general application to the classification of bifurcations for equivariant gradient maps. In particular, our results provide a complete theory when the symmetry group is discrete.

Throughout the monograph various questions arise that are of a basic nature such as the signature and linear stability question for period-$q$ points on configuration space, discrete-time conservation laws, linear normal forms, generating functions and reversibility, multiple resonant Floquet multipliers, dynamical equivalence and so forth. Each of these topics is treated independently in the appendices.

The classic theory on the bifurcation of period-$q$ points of area-preserving maps depending on a single parameter – generic bifurcation of periodic points – is due to Meyer [1970,1971] with extension to the reversible area-preserving case by Rimmer [1978,1983]. We extend the basic theory in a number of significant ways. The framework of Meyer does not extend easily to symplectic maps on $\mathbb{R}^{2n}$ with $n \geq 2$. We introduce a framework for the bifurcation of period-$q$ points based on "Lagrangian generating functions". Let $h(x, x')$ be a smooth function of $x, x' \in \mathbb{R}^n$ and suppose $\det h_{xx'}(x, x') \neq 0$ then the relations (the notation is described in Chapter 2)

$$\begin{aligned} y' &= h_{x'}(x, x') \\ y &= -h_x(x, x') \end{aligned} \tag{1.1}$$

generate, implicitly, a symplectic map $\mathbf{T} : (x, y) \mapsto (x', y')$.

The idea of Lagrangian generating functions for symplectic maps is relatively new, having been introduced in the late 1970's (Percival [1980], Aubry [1983]). However, because of their associated variational structure – an orbit of the symplectic map $\mathbf{T}$ generated by $h(x, x')$ corresponds to a critical point of the action $\mathrm{W}(x) = \sum_j h(x^j, x^{j+1})$ – Lagrangian generating functions have turned out to be extremely useful for proving existence of invariant circles, for converse KAM theory and for transport theories (reviewed in MacKay [1986] and Meiss [1991]). Most of the above results are for the area-preserving case. It is even more recent that Lagrangian generating functions have been extended to symplectic maps on $\mathbb{R}^{2n}$ , $n \geq 2$ (Bernstein & Katok [1987], MacKay, Meiss & Stark [1989], Kook & Meiss [1989], Meiss [1991]). For period-$q$ points of symplectic maps on $\mathbb{R}^{2n}$ we use Lagrangian generating functions to reduce the existence question to a problem of equivariant critical point theory. We concentrate in this work on $q \geq 3$, although in general $q$ can be any natural number.

The idea is to reduce the question of existence of bifurcating solutions to a normal form on some low (lowest) dimensional space. For symplectic maps on $\mathbb{R}^{2n}$, period-$q$ points correspond to critical points of a functional – the action functional – on an $nq$-dimensional space. We decompose the $nq$-dimensional space into an $m$-dimensional space (typically $m = 2$ or with symmetry $m = 2n$) on which the functional is "singular" and an $nq - m$ dimensional space on which the functional is nondegenerate. Because of the presence of *spatial* ($\Sigma$) and *temporal* ($\mathbb{Z}_q$) symmetries the action functional is $\Sigma \times \mathbb{Z}_q$-invariant. The Equivariant Splitting Lemma is used to decompose the action functional into a nondegenerate quadratic functional on $\mathbb{R}^{nq-m}$ and a reduced functional on $\mathbb{R}^m$ *that inherits the symmetry of the full problem.* This decomposition is basic to our analysis and with modest assumptions the idea extends to symplectic maps of arbitrary dimension, to symmetric maps with $\Sigma$ a compact Lie group and to various degeneracies of interest such as the collision of multipliers.

In other words, the existence of bifurcating period-$q$ points is in one-to-one correspondence with critical points of a $\Sigma \times \mathbb{Z}_q$-invariant functional on $\mathbb{R}^m$ and when $\Sigma = \mathrm{id}$ $m = 2$; in particular, the dimension of the reduced space, $m$, is independent of the configuration space dimension, $n$, and the sequence space dimension, $q$. In Chapter 2 a generalization of Meyer's theorem is proved on the generic bifurcation and stability of periodic points in one-parameter symplectomorphisms on $\mathbb{R}^{2n}$. In our framework the question reduces to the existence of bifurcating critical points of an $\mathbb{Z}_q$-equivariant gradient map on $\mathbb{R}^2$ dependent on a single parameter.

In subsequent chapters the theory for bifurcating period-$q$ points is extended to include

(a) the case where the symplectomorphism depends on more than one parameter and we classify all possible singularities of bifurcating period-$q$ points up to codimension 2 (with a precise definition of codimension given in Chapter 3).
(b) In 4D-symplectomorphisms, the effect of a collision of multipliers of opposite signature on bifurcating period-$q$ points (Chapters 9-10).
(c) The effect of spatial symmetry on the bifurcation of period-$q$ points (Chapter 8).

In all cases we "project" the action functional, via the Equivariant Splitting Lemma, to a reduced $\Sigma \times \mathbb{Z}_q$-equivariant gradient map on $\mathbb{R}^m$. In other words the problem can be reduced to a question in singularity theory.

Suppose we have the simplest case: $m = 2$, $\Sigma = \mathrm{id}$ and the map depends on a single parameter $\lambda$. From a singularity theory point of view, given a $\mathbb{Z}_q$-equivariant gradient map on $\mathbb{R}^2$, $\nabla_x f(x, \lambda)$, obtained from a reduction of the action functional for period-$q$ points, satisfying say

$$\nabla_x f(0,0) = 0 \quad \text{and} \quad \det \mathrm{Hess}_x f(0,0) = 0\,,$$

what is the simplest "normal form" "equivalent" to $f(x, \lambda)$? What equivalence relation is most suitable to the origin of the problem, that is bifurcating period-$q$ points ? Is the germ finitely determined ? What are all possible perturbations of $f$ (universal unfolding) ?

There are various "off the shelf" singularity theory frameworks available. For example Mather's classic contact-equivalence, without any distinguished parameter and without a gradient structure, has been used successfully by Hummel [1979] to classify $\mathbb{Z}_q$-equivariant maps on $\mathbb{R}^2$ (with an eye towards classifying singularities of periodic points of (non-symplectic) diffeomorphisms). There is catastrophe theory – right equivalence – which is natural for maps with a gradient structure. And there is singularity theory for bifurcation problems – distinguished parameter singularity theory – of Golubitsky and Schaeffer. In a similar vein the recent developments of Mond & Montaldi [1991] put special emphasis on the role of parameters and paths in parameter space.

Periodic points occur generically in one-parameter families of symplectic maps in the neighborhood of an elliptic fixed point. Therefore it is natural to include a distinguished parameter in the singularity theory framework. (There are a number of other arguments in favor of this. For example if the one-parameter represents energy of a Hamiltonian system and a bifurcation is associated with increasing energy the equivalence relation should preserve this orientation.) It is also of interest to preserve the gradient structure. Therefore the singularity theory framework we use is equivariant contact-equivalence with a distinguished parameter that preserves the gradient structure.

Here, the difficulty is that the set of such contact-equivalences does not form a nice algebraic structure; in particular, its elements depend on which gradient they are applied to. But we can circumvent the difficulties and present a new theory, in Chapter 3, in complete generality for $\Gamma$-equivariant gradient maps on $\mathbb{R}^m$ with $\Gamma$ a discrete group.

Then in Chapter 4 the singularity theory is applied to $\mathbb{Z}_q$-equivariant gradient maps on $\mathbb{R}^2$ with a distinguished parameter. This is the basic result necessary for the classification of the singularities of bifurcating period-$q$ points.

In Chapters 5 and 7 the singularities of two-parameter area-preserving maps are considered; that is, when certain coefficients, in the normal form for generic (one-parameter) period-$q$ points, go through zero the structure of the bifurcating period-$q$ points changes. In particular there are folds, symmetry-breaking bifurcations and secondary bifurcations of periodic points.

It should be noted as well that rigorous results on (linear) stability are obtained in the case where the reduced functional is on $\mathbb{R}^2$ using a generalization of the MacKay-Meiss formula (see Appendix B) and projecting onto the normal form space. In this way we are able to track the effect of degenerate bifurcations, in 2-parameter maps, on the stability configurations.

In Chapter 8 the effect of spatial symmetries on bifurcating period-$q$ points is considered. As far as we are aware this is the first work on equivariant symplectomorphisms with nontrivial spatial symmetries; in particular continuous symmetries. This is surprising as equivariant symplectic maps are quite important in applications. Simple examples giving rise to equivariant symplectic maps can be obtained by parametrically forcing some of the classical problems in mechanics such as the spherical pendulum and the rigid body. In Section 8.4 we show for example how the periodic points of an $\mathbf{O}(2)$-equivariant symplectic map are related to the dynamics of a parametrically forced spherical pendulum. A phase space of at least 4 dimensions (2D-configuration space) is necessary for non-trivial configuration space symmetries. An area-preserving map for example has only a one-dimensional configuration space: hence, the only nontrivial spatial symmetries are equivalent to $\mathbb{Z}_2$ in which case the bifurcations are not significantly different from those of a reversible area-preserving map.

Let $\mathbf{T} : (x, y) \mapsto (x', y')$ be a $\Sigma$-equivariant symplectic map with a $\Sigma$-invariant fixed point and suppose there is a bifurcation of period-$q$ points from the fixed point. Then a basic question is: how much of $\Sigma \times \mathbb{Z}_q$ is inherited by the branches of bifurcating period-$q$ points? This is a question of spontaneous symmetry breaking. A related but more abstract question is: given a spatial symmetry $\Sigma$, how many distinct families of period-$q$ points can we expect to bifurcate? We will take two approaches to this problem in Chapter 8. First, using a minimal amount of information about the $\Sigma$-equivariant symplectomorphism (spatial-temporal symmetry and gradient structure) and suitable non-degeneracy hypotheses, we use topological results – results based on $\Gamma$-length of sets of Bartsch [1992b], Bartsch & Clapp [1990] (among others) or equivariant Lusternik-Schnirelman category – to obtain lower bounds on the number of geometrically distinct branches of bifurcating period-$q$ points and their symmetry group. This idea is reminiscent of the classical "Weinstein-Moser" theory for periodic orbits of continuous-time Hamiltonian systems and generalizes to the subharmonic case a theorem of Montaldi, Roberts & Stewart [1988] on bifurcation of periodic solutions from symmetric equilibria of Hamiltonian vectorfields.

When the spatial symmetry $\Sigma$ acts absolutely irreducibly on the configuration space of the symplectic map, the results on bifurcation of period-$q$ points can be sharpened. In particular, for every isotropy subgroup $\Pi \subset \Sigma\times\mathbb{Z}_q$ with Weyl group $\mathrm{W}\Pi \stackrel{\mathrm{def}}{=} \mathrm{N}\Pi/\Pi$ (where $\mathrm{N}\Pi$ is the normalizer of $\Pi$ in $\Sigma\times\mathbb{Z}_q$) there exist *at least* $\mathrm{cat}_{\mathrm{W}\Pi}\mathrm{Fix}\,\Pi$ $\mathrm{W}\Pi$-orbits of bifurcating period-$q$ points whose symmetry group is at least $\Pi$ ($\mathrm{cat}_{\mathrm{W}\Pi}\mathrm{Fix}\,\Pi$ denotes the $\mathrm{W}\Pi$-category of the unit sphere in $\mathrm{Fix}\,\Pi$).

This result is straightforward to apply. For example, it is used in Section 8.3 to classify bifurcating period-$q$ points of $\mathbf{O}(2)$-equivariant symplectomorphisms; in particular, for each $q \geq 3$ there are 3 geometrically distinct conjugacy classes. In cases where $\dim(\mathrm{Fix}\,\Pi) = 2$ we can restrict to the fixed-point subspace and again apply the singularity theory of Chapter 4 to obtain more precise information on the branches of period-$q$ points. This program is carried out for $\Sigma = \mathbf{O}(2)$ in Section 8.3.

When a symplectomorphism has a continuous symmetry then we expect an associated conserved quantity. For symplectic maps with a Lagrangian generating function it is particularly easy to prove – via a discrete version of Noether's Theorem – that every continuous spatial symmetry generates a conserved quantity. This result is proved in Appendix E. When applied to the $\mathbf{O}(2)$-equivariant map on $\mathbb{R}^4$ is shows that, generically, every orbit of the map lies on an invariant submanifold that is equivalent to the interior of a solid torus! This leads to interesting geometric structures in the phase space. In the case of the parametrically forced spherical pendulum the conserved quantity of the symplectic map is a discrete analog of the angular momentum of the pendulum. For the $\mathbf{O}(2)$-equivariant symplectic map we give results for the period-$q$ points only, but they are quite suggestive about other interesting dynamics – invariant circles that ride on the group orbit, drift of structures transverse to the group orbit, etc. – of equivariant symplectic maps.

One of the original motivations for this study was to understand the collision of multipliers singularity in symplectic maps. The collision of multipliers of opposite signature is one of the three ways that a one-parameter family of periodic orbits in a Hamiltonian system can lose stability and it is the most difficult to analyze because it involves the bifurcation of invariant tori. Just before a collision, when the multipliers are distinct and sufficiently irrational, we expect – by KAM theory – a surrounding invariant 2-torus (in the map) but, in some sense, the 2-torus collapses to a 1-torus (an invariant circle), at the collision, that may or may not vanish after the collision. Our idea is to treat the two-parameter problem and give a complete treatment of the collision of multipliers at rational points. We will see that for $\theta = \frac{2p}{q}\pi$, and $q$ sufficiently large we get a rough idea about the structure of the irrational collision.

The collision of multipliers in symplectomorphisms is the analogue of the collision of purely imaginary eigenvalues in the vectorfield case (Meyer & Schmidt [1971], Sokol'skij [1974], van der Meer [1985], Bridges [1990,1991]). In fact, consider the linear normal form for the collision of purely imaginary eigenvalues in a Hamiltonian vectorfield,

$$\frac{d}{dt}\begin{pmatrix} q \\ p \end{pmatrix} = \mathbf{J}\mathbf{A}_0 \begin{pmatrix} q \\ p \end{pmatrix} \quad \text{with} \quad \mathbf{A}_0 = \begin{pmatrix} \mathbf{0} & -\theta\mathbf{J} \\ \theta\mathbf{J} & \epsilon\mathbf{I} \end{pmatrix}$$

with $\theta \in \mathbb{R}$ and $\epsilon = \pm 1$. Exponentiation of $\mathbf{J}\mathbf{A}_0$ then yields the linear normal form for a collision of multipliers in symplectic maps on $\mathbb{R}^4$ (see Appendix D),

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \mathbf{M}_0 \begin{pmatrix} x_n \\ y_n \end{pmatrix} \quad \text{with} \quad \mathbf{M}_0 = \exp(\mathbf{J}\mathbf{A}_0) = \begin{pmatrix} \mathbf{R}_\theta & \epsilon\mathbf{R}_\theta \\ \mathbf{0} & \mathbf{R}_\theta \end{pmatrix}$$

where $\mathbf{R}_\theta$ is the rotation matrix. Note that the value of $\theta$ (the frequency) is not particularly important in the vectorfield case but has great importance in the map case (rotation number).

An unfolding of $\mathbf{A}_0$ is given by

$$\mathbf{A}(\lambda, \alpha) = \begin{pmatrix} \alpha\mathbf{I} & -(\theta+\lambda)\mathbf{J} \\ (\theta+\lambda)\mathbf{J} & \epsilon\,\mathbf{I} \end{pmatrix}.$$

However exponentiation of $\mathbf{A}(\lambda,\alpha)$ leads to $\widehat{\mathbf{M}}(\lambda,\alpha) = \exp\big(\mathbf{J}\mathbf{A}(\lambda,\alpha)\big)$ which depends nonlinearly on $\alpha$. We prove (Proposition 9.1) that an equivalent unfolding of $\mathbf{M}_0$ in $\mathbf{Sp}(4,\mathbb{R})$ is

$$\mathbf{M}(\lambda, \alpha) = \begin{pmatrix} (1+\epsilon\,\alpha)\mathbf{R}_{\theta+\lambda} & \epsilon\mathbf{R}_{\theta+\lambda} \\ \alpha\mathbf{R}_{\theta+\lambda} & \mathbf{R}_{\theta+\lambda} \end{pmatrix}.$$

The 2-parameter matrix $\mathbf{M}(\lambda,\alpha)$ is in general position for a collision of multipliers at rational points (for a collision at irrational points $\lambda$ can be taken to be zero. On the other hand a more complete analysis of the structure of the irrational collision is obtained with the two parameters.) and is the starting point for our theory. In particular, for $(\lambda,\alpha)$ sufficiently small we study bifurcating period-$q$ points of the map $z_{n+1} = \mathbf{M}(\lambda,\alpha)\, z_n + h.o.t.$. Linear stability results for the bifurcating period-$q$ points near a collision turn out to be difficult but we make some headway on this problem in Sections 9.5 and 9.6. A sufficient condition for linear instability is obtained in Section 9.5. In Section 9.6 a linear stability theory for bifurcating period-4 points – in the unfolding of the collision at $\pm i$ – is presented. Here the interesting result is that there is always a secondary irrational, small-angle collision along one of the globally connected branches.

Of crucial importance for the study of multiplier collision is the signature of multipliers and we obtain new results on signature in configuration space for Lagrangian generating functions and this material is recorded in Appendix B.

Surprisingly there has not been much work on the bifurcation of symplectomorphisms near a collision of multipliers. The linear theory (normal form and signature theory) has been well understood for some time but the nonlinear problem is difficult because it involves the bifurcation of invariant tori in the map. Recent results of Bridges & Cushman [1993] and Bridges, Cushman & MacKay [1993] use normal form theory and Poisson reduction to obtain a geometrical picture in the phase space for the irrational collision. There are interesting numerical calculations on the bifurcations near irrational and rational collisions by Pfenniger [1985,1987]. For reversible (but non-symplectic) diffeomorphism the collision of multipliers singularity also occurs. Recent results on the collision of multipliers in one-parameter reversible maps have been reported by Sevryuk & Lahiri [1991] and Bridges, Cushman & MacKay [1993, Section 4].

Finally in Chapter 10 we consider equivariant symplectic maps with a collision of multipliers. We are mainly interested here in admissible spatial symmetries that do not prevent the basic collision in 4D-maps. The simplest additional symmetry is reversibility. However reversibility has little effect on the collision: extra $\mathbb{Z}_2$ in the normal form and extra symmetry in the sequences of period-$q$ points, but the normal form and local geometric structure in the neighborhood of the collision is similar to the non-reversible case. On the other hand non-reversible symplectomorphisms admit a larger group of spatial symmetries which in turn leads to interesting structures in the neighborhood of a collision. Admissible spatial symmetries, for symplectomorphisms on $\mathbb{R}^4$, include $\mathbf{SO}(2)$, $\mathbb{Z}_m$, $m \geq 2$, and $\mathbb{Z}_2 \times \mathbb{Z}_2$. The group $\mathbf{SO}(2)$ has the most dramatic effect on the collision (surprisingly, it does not eliminate the collision) because of the presence of a continuous symmetry and we analyze the $\mathbf{SO}(2)$-symmetric collision in some detail.

Throughout, unless otherwise mentioned (see remarks in Appendix I), the symplectic form is taken to be the standard one:

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i \quad \text{with} \quad (x,y) \in \mathbb{R}^{2n}$$

with unit symplectic operator $\mathbf{J}_{2n} = \mathbf{J} \otimes \mathbf{I}_n$. When $n = 1$ we drop the subscript:

$$\mathbf{J} \stackrel{\text{def}}{=} \mathbf{J}_{2n}\Big|_{n=1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{I} \stackrel{\text{def}}{=} \mathbf{I}_2 \,.$$

The term symplectomorphism is now becoming standard, especially in the Russian literature: the terms symplectomorphism and symplectic map are used interchangeably throughout the text. The symplectic maps and generating functions are assumed to be as smooth as necessary. Our analyses are local: we work in the neighborhood of fixed points throughout (this avoids questions of global existence of generating functions and makes it convenient to work with germs) and have found it convenient to express the phase space and configuration space variables in Euclidean coordinates – rather than action-angle variables.

Figure 1