# Binomial expansion, power series, limits, approximations, Fourier series

*Notice: this material must not be used as a substitute for attending the lectures*

# 1 Binomial expansion

We know that

$$\begin{aligned}(a+b)^1 &= a+b\\(a+b)^2 &= a^2+2ab+b^2\\(a+b)^3 &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

The question is (at this stage): what about $(a+b)^n$ where $n$ is any positive integer?

## 1.1 Pascal's triangle

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 1 & & 1 & & & & & \\ & & & & 1 & & 2 & & 1 & & & & \\ & & & 1 & & 3 & & 3 & & 1 & & & \\ & & 1 & & 4 & & 6 & & 4 & & 1 & & \\ & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\ 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1\end{array}$$

To expand $(a+b)^n$ we look for the row starting with 1 and $n$.

## 1.2 Example

Let's expand $(a+b)^3$. The row in Pascal's triangle starting with 1 and 3 is

$$1 \quad 3 \quad 3 \quad 1$$

Therefore the expansion of $(a+b)^3$ is

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

## 1.3 Example

Let's expand $(a+b)^6$.

The row starting with 1 and 6 in Pascal's triangle is the row

$$1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1$$

This means that the expansion of $(a+b)^6$ is

$$(a+b)^6 = a^6+6a^5b+15a^4b^2+20a^3b^3+15a^2b^4+6ab^5+b^6$$

## 1.4 Factorial notation

The **factorial** $n!$ of a positive integer $n$ is defined by

$$n! = n(n-1)(n-2)\cdots(3)(2)(1)$$

so for example

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

and

$$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$$

We work with the convention that

$$1! = 1 \quad \text{and} \quad 0! = 1$$

Expressions involving factorials can often be simplified as shown in the example below:

$$\frac{8!}{5!\,3!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(5 \times 4 \times 3 \times 2 \times 1)(3 \times 2 \times 1)} = \frac{8 \times 7 \times 6}{6} = 56$$

## 1.5 Binomial theorem

Pascal's triangle can be difficult to use if the exponent is very high. In such cases the following **binomial theorem** is usually better. This states that if $n$ is a **positive integer** then

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + b^n$$

An important particular case is when $a = 1$ and $b = x$ giving

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots + x^n \qquad (1.1)$$

which, like the previous result, holds for positive integers $n$.

In the binomial theorem, the general term has the form $a^{n-m}b^m$ with coefficient

$$\frac{n(n-1)(n-2)\cdots(n-(m-1))}{m!}$$

which equals

$$\frac{n(n-1)(n-2)\cdots(n-(m-1))(n-m)!}{m!\,(n-m)!}$$

or

$$\frac{n!}{m!\,(n-m)!} \qquad \text{often denoted} \quad \binom{n}{m}$$

In terms of the notation introduced above, the binomial theorem can be written as

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n = \sum_{i=0}^{n}\binom{n}{i}a^{n-i}\,b^i$$

## 1.6 Example

Expand $\left(2+\frac{x}{3}\right)^4$.

*Solution.* Using the binomial theorem:

$$\begin{aligned}\left(2+\frac{x}{3}\right)^4 &= 2^4+(4)(2^3)(\frac{x}{3})+\frac{(4)(3)}{2!}(2^2)(\frac{x}{3})^2+\frac{(4)(3)(2)}{3!}(2)(\frac{x}{3})^3+\frac{(4)(3)(2)(1)}{4!}(\frac{x}{3})^4\\ &= 16+\frac{32}{3}x+\frac{8}{3}x^2+\frac{8}{27}x^3+\frac{1}{81}x^4.\end{aligned}$$

## 1.7 Example

Expand $\left(1+\frac{x}{3}\right)^{15}$ up to and including the term in $x^3$.

*Solution.* By the binomial theorem:

$$\begin{aligned}\left(1+\frac{x}{3}\right)^{15} &= 1+15(\frac{x}{3})+\frac{(15)(14)}{2!}(\frac{x}{3})^2+\frac{(15)(14)(13)}{3!}(\frac{x}{3})^3+\cdots\\ &= 1+5x+\frac{35}{3}x^2+\frac{455}{27}x^3+\cdots\end{aligned}$$

## 1.8 Example

Expand $(1-x)^3(2+x)^6$ up to and including the term in $x^2$.

*Solution.*

$$\begin{aligned}(1-x)^3(2+x)^6 &= (1-x)^3\left(2^6+(6)(2^5)x+\frac{(6)(5)}{2!}(2^4)x^2+\cdots\right)\\ &= \left(1+3(-x)+\frac{(3)(2)}{2!}(-x)^2+\underbrace{(-x)^3}_{\text{redundant}}\right)(64+192x+240x^2+\cdots)\\ &= (1-3x+3x^2-x^3)(64+192x+240x^2+\cdots)\\ &= 64+(192-(64)(3))x+(3(64)-3(192)+240)x^2\\ &= 64-144x^2+\cdots\end{aligned}$$

## 1.9 Powers that are NOT positive integers

The binomial expansion as discussed up to now is for the case when the exponent is a positive integer only.

For the case when the number $n$ is **not** a positive integer the binomial theorem becomes, for $-1<x<1$,

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots \tag{1.2}$$

This might look the same as the binomial expansion given by expression (1.1), but let us make the following important distinctions between (1.1) and (1.2):

- the expansion for positive integer powers (expansion (1.1)) *terminates*, i.e. it has only a finite number of terms. However, for powers that are not positive integers the series (1.2) is an infinite series that goes on forever.

- it can be mathematically proven that the series (1.2) is valid only for $-1 < x < 1$.

- expression (1.2) cannot be applied to something of the form $(a+x)^n$. Such an expression must first be rewritten as follows:

$$(a+x)^n = \left(a\left(1+\frac{x}{a}\right)\right)^n = a^n \underbrace{\left(1+\frac{x}{a}\right)^n}_{\text{apply binomial to this}}$$

## 1.10 Example

Expand $\sqrt{1+2x}$ and state what values of $x$ the series is valid.

*Solution.*

$$\begin{aligned}\sqrt{1+2x} &= (1+2x)^{1/2}\\ &= 1+\frac{1}{2}(2x)+\frac{(\frac{1}{2})(-\frac{1}{2})}{2!}(2x)^2+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}(2x)^3+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!}(2x)^4+\cdots\\ &= 1+x-\frac{1}{2}x^2+\frac{1}{2}x^3-\frac{5}{8}x^4+\cdots\end{aligned}$$

This series is valid when $-1 < 2x < 1$. i.e. when $-\frac{1}{2} < x < \frac{1}{2}$.

## 1.11 Example

Expand $\left(1-\frac{x}{2}\right)^{-5}$. For what values of $x$ is the expansion valid?

*Solution.*

$$\begin{aligned}\left(1-\frac{x}{2}\right)^{-5} &= 1+(-5)\left(-\frac{x}{2}\right)+\frac{(-5)(-6)}{2!}\left(-\frac{x}{2}\right)^2+\frac{(-5)(-6)(-7)}{3!}\left(-\frac{x}{2}\right)^3+\cdots\\ &= 1+\frac{5}{2}x+\frac{15}{4}x^2+\frac{35}{8}x^3+\cdots\end{aligned}$$

This is valid when $-1 < -\frac{x}{2} < 1$, i.e. when $-2 < x < 2$.

## 1.12 Example

Expand $(3+x)^{-\frac{1}{2}}$.

*Solution.* Remember that when the power is not a positive integer your expression has to be of the form $(1+\text{something})^{\text{power}}$. Deal with this as follows:

$$(3+x)^{-\frac{1}{2}} = \left(3\left(1+\frac{x}{3}\right)\right)^{-\frac{1}{2}} = 3^{-\frac{1}{2}}\underbrace{\left(1+\frac{x}{3}\right)^{-\frac{1}{2}}}_{\text{expand this}}$$

$$= 3^{-\frac{1}{2}}\left(1+(-\frac{1}{2})(\frac{x}{3})+\frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}(\frac{x}{3})^2+\cdots\right)$$
$$= \frac{1}{\sqrt{3}}\left(1-\frac{x}{6}+\frac{x^2}{24}+\cdots\right)$$

This is valid when $-1 < x/3 < 1$, i.e. when $-3 < x < 3$.

## 1.13 Example

Find expansions for $\left(1+\frac{1}{x}\right)^{1/2}$ for the cases (i) $|x| > 1$ and (ii) $0 < x < 1$.

*Solution.* the following calculation produces an expansion which will be valid when $1/|x| < 1$, i.e. $|x| > 1$:

$$\left(1+\frac{1}{x}\right)^{1/2} = 1+\frac{1}{2}\left(\frac{1}{x}\right)+\frac{(\frac{1}{2})(-\frac{1}{2})}{2!}\left(\frac{1}{x}\right)^2+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}\left(\frac{1}{x}\right)^3+\cdots$$
$$= 1+\frac{1}{2x}-\frac{1}{8x^2}+\frac{1}{16x^3}+\cdots$$

valid for $|x| > 1$.

The above expansion is no good if $|x| < 1$. For this case the following trick produces a valid expansion:

$$\left(1+\frac{1}{x}\right)^{1/2} = \left(\frac{x+1}{x}\right)^{1/2} = \frac{1}{x^{1/2}}\underbrace{(1+x)^{1/2}}_{\text{expand this}}$$
$$= \frac{1}{x^{1/2}}\left(1+\frac{1}{2}x+\frac{(\frac{1}{2})(-\frac{1}{2})}{2!}x^2+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3+\cdots\right)$$
$$= \frac{1}{x^{1/2}}\left(1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\cdots\right)$$
$$= \frac{1}{x^{1/2}}+\frac{1}{2}x^{1/2}-\frac{1}{8}x^{3/2}+\frac{1}{16}x^{5/2}+\cdots$$

Note that this is actually defined only for $0 < x < 1$.

## 1.14 Example

Expand $\frac{(1+x)^2}{(1-x/2)^3}$ up to and including the term in $x^2$.

*Solution.*

$$\frac{(1+x)^2}{(1-x/2)^3} = (1+x)^2\left(1-\frac{x}{2}\right)^{-3}$$
$$= (1+2x+x^2)\left(1+(-3)\left(-\frac{x}{2}\right)+\frac{(-3)(-4)}{2!}\left(-\frac{x}{2}\right)^2+\cdots\right)$$
$$= (1+2x+x^2)\left(1+\frac{3x}{2}+\frac{3x^2}{2}+\cdots\right)$$

$$
\begin{aligned}
&= 1+\left(\frac{3}{2}+2\right)x+\left(1+2\left(\frac{3}{2}\right)+\frac{3}{2}\right)x^2+\cdots \\
&= 1+\frac{7x}{2}+\frac{11x^2}{2}\cdots
\end{aligned}
$$

# 2 Taylor and Maclaurin series

## 2.1 Taylor series

The idea is to expand a function $f(x)$ about a point $a$ in the form of a sum of powers of $(x-a)$, i.e. to form a series of the form

$$f(x)=a_0+a_1(x-a)+a_2(x-a)^2+a_3(x-a)^3+\cdots=\sum_{n=0}^{\infty}a_n(x-a)^n \tag{2.3}$$

we want to know the coefficients $a_n$, $n=0,1,2,\ldots$ in the above expansion.

If we differentiate expression (2.3) again and again, we get the following expressions for the first, second, third, etc derivatives of $f(x)$:

$$
\begin{aligned}
f'(x) &= a_1+2a_2(x-a)+3a_3(x-a)^2+4a_4(x-a)^3+\cdots \\
f''(x) &= 2a_2+(3)(2)a_3(x-a)+(4)(3)a_4(x-a)^2+\cdots \\
f'''(x) &= (3)(2)a_3+(4)(3)(2)a_4(x-a)+\cdots \\
\vdots \quad & \quad \vdots
\end{aligned}
$$

Putting $x=a$ in these expressions gives

$$
\begin{aligned}
f'(a) &= a_1 && \Rightarrow\ a_1=f'(a) \\
f''(a) &= 2a_2 && \Rightarrow\ a_2=\frac{1}{2}f''(a) \\
f'''(a) &= (3)(2)a_3 && \Rightarrow\ a_3=\frac{1}{(2)(3)}f'''(a)
\end{aligned}
$$

Spotting the pattern, we see that the general formula for the coefficient $a_n$ will be

$$a_n=\frac{1}{n!}f^{(n)}(a)$$

where $f^{(n)}(a)$ means the $n$th derivative of $f(x)$, evaluated at the value $x=a$.

This gives us what we call the **Taylor expansion** of a function $f(x)$ valid for values of $x$ near to $a$:

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\cdots \tag{2.4}$$

The series carries on to infinity, and has general term $\frac{(x-a)^n}{n!}f^{(n)}(a)$.

Taylor's expansion, and the related Maclaurin expansion discussed below, are used in approximations. In practice usually only the first few terms in the series are kept and the rest are discarded. The idea is that the resulting truncated expansion should provide a good approximation to the function $f(x)$ for values of $x$ close to the particular value $a$. The more terms we keep, the better the approximation.

## 2.2 Maclaurin series

There is also the **Maclaurin expansion**, which is just the Taylor expansion in the particular case when $a = 0$, i.e.

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots \tag{2.5}$$

or, in summation notation

$$f(x) = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!}$$

Not all functions have Taylor or Maclaurin expansions but most do.

## 2.3 Example

Let us find the Maclaurin series of $e^x$.

*Solution.* Let $f(x) = e^x$.

Then $f(0) = 1$.

Also $f'(x) = e^x$ so $f'(0) = 1$.

$f''(x) = e^x$ so $f''(0) = 1$. Clearly in this particular example $f^{(n)}(0) = 1$ for all $n = 1, 2, 3, \ldots$. Putting these values for $f(0)$, $f'(0)$, $f''(0)$, etc, into (2.5) gives us the Maclaurin series for the particular function $f(x) = e^x$, namely

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{2.6}$$

or, in summation notation, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

## 2.4 Example

Deduce the Maclaurin series of $e^{5x}$ from that for $e^x$.

*Solution.* Just replace every $x$ by $5x$ in expression (2.6) above to get

$$\begin{aligned} e^{5x} &= 1 + 5x + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + \cdots \\ &= 1 + 5x + \frac{25x^2}{2} + \frac{125x^3}{6} + \cdots \end{aligned}$$

## 2.5 Example

Find the Maclaurin series of $\cos x$.

*Solution.* Let $f(x) = \cos x$.

Then $f(0) = 1$.

Also $f'(x) = -\sin x$ so $f'(0) = 0$.

$f''(x) = -\cos x$ so $f''(0) = -1$.

$f'''(x) = \sin x$ so $f'''(0) = 0$.

$f''''(x) = \cos x$ so $f''''(0) = 1$.
$f'''''(x) = -\sin x$ so $f'''''(0) = 0$.

We see the pattern emerging. The values $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$, etc, cycle through the values $1, 0, -1, 0, 1, 0, -1, 0, \ldots$. Putting these values into the general Maclaurin expansion (2.5) gives the Maclaurin expansion for the function $\cos x$, namely

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$$

or, in summation notation,

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Similarly, it can be shown that the Maclaurin expansion of $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

## 2.6 Example

Find the **Taylor** series of the function $f(x) = 1/x$ about $x = 2$.

*Solution.* We are asked for a **Taylor** series here, not the Maclaurin one. The relevant formula is therefore (2.4) in the case when $a = 2$. So we need to work out the values $f(2)$, $f'(2)$, $f''(2)$, etc. We do this next:

$f(2) = \frac{1}{2}$.
$f'(x) = -\frac{1}{x^2}$ so $f'(2) = -\frac{1}{4}$.
$f''(x) = \frac{2}{x^3}$ so $f''(2) = \frac{1}{4}$.
$f'''(x) = -\frac{6}{x^4}$ so $f'''(2) = -\frac{3}{8}$,
and so on. The Taylor series about the value $x = 2$ is

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \cdots$$

which becomes, since $f(x) = 1/x$,

$$\frac{1}{x} = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{1}{8}(x-2)^2 - \frac{1}{16}(x-2)^3 + \cdots$$

What this means, is that the first few terms of the above series expansion will constitute a good approximation to $1/x$ for values of $x$ close to 2.

Note that the function $f(x) = 1/x$ does not have a Taylor series expansion about the point $x = 0$. This is because this function goes to infinity as $x \to 0$, so we could hardly expect the function to have an approximation for small values of $x$ as a series of powers of $x$. Had we attempted to find $f(0)$, $f'(0)$, $f''(0)$, etc, they would all turn out to be infinity.

## 2.7 Example

Find the first three non-zero terms of the Maclaurin series of $e^{-2x}\sin x$.

*Solution.* One way to do this would be to write down the Maclaurin series for $e^{-2x}$ (which can be inferred from the one for $e^x$ by replacing every $x$ by $-2x$) and the series for $\sin x$ and then multiplying the series together and expanding out. The approach below is a direct approach not requiring such advance knowledge of the two separate Maclaurin expansions.

Let $f(x) = e^{-2x}\sin x$. Then $f(0) = 0$.

$f'(x) = e^{-2x}\cos x - 2e^{-2x}\sin x$ so $f'(0) = 1$. Differentiating again

$$\begin{aligned} f''(x) &= e^{-2x}(-\sin x) - 2e^{-2x}\cos x - 2(e^{-2x}\cos x - 2e^{-2x}\sin x) \\ &= 3e^{-2x}\sin x - 4e^{-2x}\cos x \end{aligned}$$

and

$$f'''(x) = 3(e^{-2x}\cos x - 2e^{-2x}\sin x) - 4(-e^{-2x}\sin x - 2e^{-2x}\cos x)$$

From these expressions we get $f''(0) = -4$ and $f'''(0) = 11$. Putting these values into the general Maclaurin series (2.5) gives the following expression for our particular function $f(x) = e^{-2x}\sin x$:

$$e^{-2x}\sin x = x - 2x^2 + \frac{11x^3}{6} + \cdots$$

which will constitute a good approximation to $e^{-2x}\sin x$ provided $x$ is reasonably small.

## 2.8 Example

Find the binomial expansion of $(1-x^2)^{-1/2}$ and deduce from it a power series expansion for $\sin^{-1} x$.

*Solution.* First we find the expansion of $(1+x)^{-1/2}$.

$$\begin{aligned} (1+x)^{-1/2} &= 1 + (-\frac{1}{2})x + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}x^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}x^3 + \cdots \\ &= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots \end{aligned}$$

In the above, we now replace every $x$ by $-x^2$ to deduce that

$$\begin{aligned} (1-x^2)^{-1/2} &= 1 - \frac{1}{2}(-x^2) + \frac{3}{8}(-x^2)^2 - \frac{5}{16}(-x^2)^3 + \cdots \\ &= 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \frac{5}{16}x^6 + \cdots \end{aligned}$$

Now

$$\sin^{-1} x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$$

$$
\begin{aligned}
&= \int_0^x (1-t^2)^{-1/2}\,dt \\
&= \int_0^x \left(1 + \frac{1}{2}t^2 + \frac{3}{8}t^4 + \frac{5}{16}t^6 + \cdots\right) dt \\
&= x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \cdots
\end{aligned}
$$

# 3 Applications to working out limits

The notation

$$\lim_{x\to a} f(x)$$

means the value (if any) that $f(x)$ approaches, when $x$ approaches $a$. The word "lim" means **limit**.

## 3.1 Important issues to do with limits

Two trivial examples of working out limits would be

$$\lim_{x\to 2}(x^2-3) = 1, \qquad \lim_{x\to 0}\cos x = 1$$

In the above examples we can just put the value in. But in many situations we cannot do this because we end up with the mathematically meaningless expression $\frac{0}{0}$ which could be anything.

For example, let's work out

$$\lim_{x\to 2}\frac{x^2-4}{x-2}$$

In this example we cannot put $x = 2$ into the expression otherwise we get $\frac{0}{0}$ which could be anything. But we can simplify the expression by factorising and cancelling factors to get

$$\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4$$

Similarly, let's work out

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}$$

Again we cannot just put $x = 1$ into this expression or we would get $\frac{0}{0}$. But we can factorise and simplify as follows:

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x} = \lim_{x\to 1}\frac{(x-1)(x+2)}{x(x-1)} = \lim_{x\to 1}\frac{x+2}{x} = 3.$$

It is not always possible to work out limits simply by looking for factors and simplifying as in the above examples. We now want to add binomial expansion and Taylor/Maclaurin series to our list of methods for working out limits.

## 3.2 Example

Let's work out

$$\lim_{x\to 0}\frac{(1+x/2)^{5/7}-1}{x}$$

Again, we cannot put $x=0$ into this expression as it stands. But we can use binomial expansion, as follows;

$$\begin{aligned}\frac{(1+x/2)^{5/7}-1}{x} &= \frac{\left[1+(\frac{5}{7})(\frac{x}{2})+\frac{(\frac{5}{7})(-\frac{2}{7})}{2!}(\frac{x}{2})^2+\cdots\right]-1}{x}\\ &= \frac{\frac{5}{14}x-\frac{5}{196}x^2+\cdots}{x}\\ &= \frac{5}{14}-\frac{5}{196}x+\cdots\end{aligned}$$

We can let $x\to 0$ in the above expression to deduce that

$$\lim_{x\to 0}\frac{(1+x/2)^{5/7}-1}{x}=\frac{5}{14}$$

## 3.3 Example

Let's work out

$$\lim_{x\to 0}\frac{\sin x}{x} \qquad \text{and} \qquad \lim_{x\to 0}\frac{\sin 2x}{x}$$

*Solution.* We mentioned earlier that

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots$$

Hence

$$\frac{\sin x}{x}=1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots$$

We can let $x\to 0$ in this to deduce that

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

From the Maclaurin expansion for $\sin x$ given above, we can deduce the expansion for $\sin 2x$ to be

$$\begin{aligned}\sin 2x &= 2x-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}-\cdots\\ &= 2x-\frac{4x^3}{3}+\frac{32x^5}{120}-\cdots\end{aligned}$$

Hence

$$\frac{\sin 2x}{x}=2-\frac{4x^2}{3}+\cdots$$

Letting $x\to 0$ we deduce that

$$\lim_{x\to 0}\frac{\sin 2x}{x}=2$$

It is in fact a general result that $\lim_{x\to 0}\frac{\sin kx}{x}=k$ for any constant $k$.

## 3.4 Example

Find

$$\lim_{x\to 0}\frac{\sin^2 x - x^2\cos x}{x^4}$$

*Solution.* Recall that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

and

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Squaring the formula for $\sin x$ gives

$$\begin{aligned}
\sin^2 x &= \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right) \\
&= x^2 - \frac{x^4}{6} + (\text{something})\, x^6 - \frac{x^4}{6} + (\text{something})\, x^6 \\
&= x^2 - \frac{x^4}{3} + (\text{something})\, x^6
\end{aligned}$$

Hence, using also the expansion for $\cos x$ given above, we have

$$\begin{aligned}
\frac{\sin^2 x - x^2\cos x}{x^4} &= \frac{\left(x^2 - \frac{x^4}{3} + (\text{something})\, x^6 + \cdots\right) - x^2\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right)}{x^4} \\
&= \frac{\frac{1}{6}x^4 + (\text{something})\, x^6 + \text{even higher powers of } x}{x^4} \\
&= \frac{1}{6} + (\text{something})\, x^2 + \cdots
\end{aligned}$$

Let $x \to 0$ in the above to get

$$\lim_{x\to 0}\frac{\sin^2 x - x^2\cos x}{x^4} = \frac{1}{6}$$

## 3.5 Example

Find

$$\lim_{x\to\infty} x(e^{-1/x} - 1)$$

*Solution.* To deal with $x$ going to infinity, we shall let $y = 1/x$ and let $y \to 0$. This gives

$$\begin{aligned}
\lim_{x\to\infty} x(e^{-1/x} - 1) &= \lim_{y\to 0}\frac{1}{y}(e^{-y} - 1) \\
&= \lim_{y\to 0}\frac{1}{y}\left(\left\{1 + (-y) + \frac{(-y)^2}{2!} + \cdots\right\} - 1\right) \\
&= \lim_{y\to 0}\left(-1 + \frac{y}{2!} + \cdots\right) \\
&= -1
\end{aligned}$$

where we have used the Maclaurin expansion for the exponential, given by (2.6).

# 4 L'Hopital's rule

Another way of working out a limit when in a $\frac{0}{0}$ situation is the following result:

$$\text{if } f(a) = 0 \text{ and } g(a) = 0 \text{ then} \qquad \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

The above result is called **L'Hopital's rule**.

It is absolutely crucial to check the condition $f(a) = 0$ and $g(a) = 0$ before using the rule, because it does not work otherwise.

## 4.1 Example

$$\begin{aligned} &\lim_{x \to 0} \frac{3x - \sin x}{x} && \text{would be } \tfrac{0}{0} \text{ if we put } x = 0 \text{ in, so use L'Hopital} \\ &= \lim_{x \to 0} \frac{3 - \cos x}{1} && \text{no longer } \tfrac{0}{0} \\ &= \frac{3 - \cos 0}{1} \\ &= 2 \end{aligned}$$

## 4.2 Example

$$\begin{aligned} &\lim_{x \to 0} \frac{1 - \cos x}{x + x^2} && \text{would be } \tfrac{0}{0} \text{ if we put } x = 0 \text{ in, so use L'Hopital} \\ &= \lim_{x \to 0} \frac{\sin x}{1 + 2x} && \text{no longer } \tfrac{0}{0} \\ &= \frac{0}{1} \\ &= 0 \end{aligned}$$

## 4.3 Example

$$\lim_{x \to 2} \frac{x - 2}{x^2 - 4} = \lim_{x \to 2} \frac{1}{2x} = \frac{1}{4}$$

Sometimes we have to apply L'Hopital's rule more than once to get an answer, as the next example illustrates:

### 4.4 Example

$$\begin{aligned}
&\lim_{x\to 0}\frac{x-\sin x}{x^3} && \tfrac{0}{0} \text{ so use L'Hopital}\\
&=\lim_{x\to 0}\frac{1-\cos x}{3x^2} && \text{still } \tfrac{0}{0} \text{ so use L'Hopital again}\\
&=\lim_{x\to 0}\frac{\sin x}{6x} && \text{still } \tfrac{0}{0} \text{ so use L'Hopital again}\\
&=\lim_{x\to 0}\frac{\cos x}{6} && \text{no longer } \tfrac{0}{0}\\
&=\frac{1}{6}
\end{aligned}$$

### 4.5 Example

$$\begin{aligned}
&\lim_{x\to 0}\frac{\ln\cos x}{\ln\cos 3x} && \tfrac{0}{0} \text{ so use L'Hopital}\\
&=\lim_{x\to 0}\frac{\left(-\frac{\sin x}{\cos x}\right)}{\left(-\frac{3\sin 3x}{\cos 3x}\right)} && \text{now simplify this}\\
&=\lim_{x\to 0}\frac{\tan x}{3\tan 3x} && \text{still } \tfrac{0}{0} \text{ so use L'Hopital again}\\
&=\lim_{x\to 0}\frac{\sec^2 x}{9\sec^2 3x} && \text{no longer } \tfrac{0}{0}\\
&=\frac{1}{9}
\end{aligned}$$

## 5 Fourier Series

A **Fourier Series** is an expansion of a periodic function as an infinite sum of sines and cosines.

Simple examples of periodic functions (other than sin and cos) are the square wave and sawtooth functions. An example of a square wave function (of period 4 in this particular case) is the periodic function

$$f(t) = \begin{cases} -1, & -2 < t < 0, \\ 1, & 0 < t < 2, \end{cases}$$

with $f(t+4) = f(t)$.

An example of a sawtooth function of period $2\pi$ would be the periodic function of period $2\pi$ such that $f(t) = t$ for $t \in (-\pi, \pi)$. Since this function has period $2\pi$ we might suppose that it has an expansion in terms of the functions $\cos t$, $\cos 2t$, $\cos 3t, \ldots$ and the functions $\sin t$, $\sin 2t$, $\sin 3t, \ldots$ since these functions also have period $2\pi$. Such an expansion does indeed exist and in fact any periodic function of period $2\pi$ has an expansion in terms of these trigonometric functions.

If the period is $T$ rather than $2\pi$ this is no particular problem. All we have to do is modify the period of the cos and sin functions we work with, i.e. we instead seek an expansion in terms of the functions $\cos\frac{2n\pi t}{T}$ and $\sin\frac{2n\pi t}{T}$ for $n = 1, 2, 3\ldots$, rather than $\cos nt$ and $\sin nt$. Letting $f(t)$ be a $T_0$ periodic function, this expansion, called the **Fourier series** of $f(t)$, turns out to be

$$f(t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi t}{T_0} + b_n \sin\frac{2n\pi t}{T_0}\right) \tag{5.7}$$

where

$$a_n = \frac{2}{T_0}\int_0^{T_0} f(t)\cos\frac{2n\pi t}{T_0}\,dt, \qquad n = 0, 1, 2, 3, \ldots \tag{5.8}$$

$$b_n = \frac{2}{T_0}\int_0^{T_0} f(t)\sin\frac{2n\pi t}{T_0}\,dt, \qquad n = 1, 2, 3, \ldots \tag{5.9}$$

A number of **important points** need to be made:

- When working out the integrals in (5.8,5.9) you can in fact use **any** interval of length $T_0$. As a consequence, the alternative formulae:

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\cos\frac{2n\pi t}{T_0}\,dt, \qquad n = 0, 1, 2, 3, \ldots \tag{5.10}$$

$$b_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\sin\frac{2n\pi t}{T_0}\,dt, \qquad n = 1, 2, 3, \ldots \tag{5.11}$$

  will work just as well.

- to work out $a_0$ in (5.7) you use the $a_n$ formula (either (5.8) or (5.10)) with $n = 0$. You will sometimes find that the $n = 0$ case needs to be dealt with separately from the other $a_n$ coefficients due to division by zero problems.

- the quantity $T_0$ is the **period** of the wave so the frequency would be $1/T_0$, usually measured in cycles per second. It is, however, more usual to define the frequency to be the quantity $\omega_0$ defined by

$$\omega_0 = \frac{2\pi}{T_0} \qquad \text{rather than } \tfrac{1}{T_0}$$

- often we want to work out the Fourier series of a periodic function that contains points of discontinuity (the abovementioned square wave and sawtooth functions being examples). It is known that, at a point of discontinuity (at $x = a$, say) the Fourier series of the function converges to

$$\tfrac{1}{2}(f(a+) + f(a-))$$

  rather than to $f(a)$. This applies regardless of how $f(t)$ is defined (if it is defined at all) at the point $a$ itself. In the above formula the notation $f(a+)$ means the value just to the right of the discontinuity and $f(a-)$ means the value to the left. More formally, $f(a+)$ is the limit of $f(a + h)$ as $h$ tends to zero from above, and $f(a-)$ is the limit of $f(a - h)$ as $h$ tends to zero from above.

## 5.1 Example

Let

$$f(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 < t < \pi \end{cases}$$

with $f(t + 2\pi) = f(t)$. Find the Fourier series of $f(t)$.

*Solution.* In this case the period $T_0$ is given by $T_0 = 2\pi$. Let us find $a_n$ first. The following formulae will be useful

$$\cos n\pi = (-1)^n \quad \text{and} \quad \sin n\pi = 0, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

We have

$$\begin{aligned} a_n &= \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} f(t) \cos \frac{2n\pi t}{T_0}\, dt \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt\, dt \\ &= \frac{1}{\pi} \int_{-\pi}^{0} (-1) \cos nt\, dt + \frac{1}{\pi} \int_{0}^{\pi} \cos nt\, dt \\ &= \frac{1}{\pi} \left[ \frac{-\sin nt}{n} \right]_{-\pi}^{0} + \frac{1}{\pi} \left[ \frac{\sin nt}{n} \right]_{0}^{\pi} \\ a_n &= 0 \end{aligned}$$

Because of the $n$ in the denominator of the above calculations we need to find $a_0$ separately, but it turns out also to be zero. *We would warn you in advance, however, that in plenty of other situations a separate calculation for $a_0$ is absolutely essential for a correct Fourier series.*

Now let's find $b_n$. we have

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt\, dt \\ &= \frac{1}{\pi} \left[ \int_{-\pi}^{0} (-1) \sin nt\, dt + \int_{0}^{\pi} \sin nt\, dt \right] \\ &= \frac{1}{\pi} \left[ \left[ \frac{\cos nt}{n} \right]_{-\pi}^{0} + \left[ \frac{-\cos nt}{n} \right]_{0}^{\pi} \right] \\ &= \frac{1}{\pi n} \left( 1 - \cos(-n\pi) + (-\cos n\pi + 1) \right) \\ &= \frac{1}{\pi n} (2 - 2(-1)^n) \end{aligned}$$

and so

$$b_n = \frac{2}{\pi n} (1 - (-1)^n)$$

With all the $a_n$, $n = 0, 1, 2, \ldots$, equal to zero, and also recalling that the period $T_0 = 2\pi$, the Fourier series becomes

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty} b_n \sin nt \\ &= b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t + \cdots \end{aligned}$$

i.e.

$$f(t) = \frac{4}{\pi}\sin t + \frac{4}{3\pi}\sin 3t + \frac{4}{5\pi}\sin 5t + \cdots$$

# 6 Even and odd functions

If a function is an even function or an odd function then certain simplifications are possible in the calculations required for computing the Fourier series. But note that plenty of functions are **neither even nor odd**, eg $f(t) = t^2 + t$.

## 6.1 Even functions

$f(t)$ is said to be an **even function** if $f(-t) = f(t)$. This means the graph is symmetrical about the $y$-axis.

Examples of even functions are $f(t) = \text{constant}$, $f(t) = t^2$, $f(t) = t^4$, $f(t) = t^6, \ldots$ (all even powers of $t$); also $f(t) = \cos t$ and $f(t) = \cosh t$.

An even function has the important property that

$$\int_{-a}^{a} f(t)\,dt = 2\int_{0}^{a} f(t)\,dt$$

## 6.2 Odd functions

$f(t)$ is said to be an **odd function** if $f(-t) = -f(t)$. This means the graph has $180^o$ rotational symmetry about the origin.

Examples of odd functions are $f(t) = t$, $f(t) = t^3$, $f(t) = t^5, \ldots$ (all odd powers of $t$); also $f(t) = \sin t$ and $f(t) = \sinh t$.

An odd function has the important property that

$$\int_{-a}^{a} f(t)\,dt = 0$$

## 6.3 Useful rules of even and odd functions

$$\begin{aligned} \text{even} \times \text{even} &= \text{even} \\ \text{even} \times \text{odd} &= \text{odd} \\ \text{odd} \times \text{even} &= \text{odd} \\ \text{odd} \times \text{odd} &= \text{even} \end{aligned}$$

## 6.4 Fourier series of an even function

Suppose that $f(t)$ is an even function and we want it's Fourier series. Since $\sin t$ is an odd function we might anticipate that the Fourier series of an even function will contain no sine terms. We shall show that this is indeed the case. With $f(t)$ being even the $b_n$ Fourier coefficient is given by

$$\begin{aligned} b_n &= \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} \underbrace{f(t)}_{\text{even}} \underbrace{\sin\frac{2n\pi t}{T_0}}_{\text{odd}}\, dt \\ &= \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} (\text{something odd})\, dt \\ &= 0 \end{aligned}$$

So if $f(t)$ is even we can declare from the outset that the $b_n$ terms are all zero, and we only need to work out $a_n$, $n = 0, 1, 2, \ldots$, a separate calculation often being needed for $a_0$. With $f(t)$ being even we get an alternative formula for $a_n$ as follows:

$$\begin{aligned} a_n &= \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} \underbrace{f(t)}_{\text{even}} \underbrace{\cos\frac{2n\pi t}{T_0}}_{\text{even}}\, dt \\ &= \frac{4}{T_0}\int_{0}^{T_0/2} f(t)\cos\frac{2n\pi t}{T_0}\, dt \end{aligned}$$

This can save us time and effort.

## 6.5 Fourier series of an odd function

If $f(t)$ is odd then the $a_n$ coefficients (including $a_0$) are zero because

$$\begin{aligned} a_n &= \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} \underbrace{f(t)}_{\text{odd}} \underbrace{\cos\frac{2n\pi t}{T_0}}_{\text{even}}\, dt \\ &= \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} (\text{something odd})\, dt \\ &= 0 \end{aligned}$$

Thus the Fourier series of an odd function contains only sine terms. Moreover, calculation of the $b_n$ coefficients of these sine terms can be simplified by exploiting the oddness property.

## 6.6 Example

Find the Fourier series of the sawtooth function given by

$$f(t) = t \quad \text{when} \quad -2 < t < 2$$

with $f(t+4) = f(t)$ for all $t$ (i.e. the function has period 4).

*Solution.* This function is **odd**. Its graph has $180^o$ rotational symmetry about the origin. Since it is odd, we can immediately say that $a_n = 0$ for all $n$ (including $n = 0$) and we only need to calculate $b_n$.

Also note that since the period is 4, we have in this case $T_0 = 4$. We now find $b_n$:

$$\begin{aligned}
b_n &= \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} f(t)\sin\frac{2n\pi t}{T_0}\,dt \\
&= \frac{2}{4}\int_{-2}^{2} f(t)\sin\frac{2n\pi t}{4}\,dt \\
&= \frac{1}{2}\int_{-2}^{2} \underbrace{t}_{\text{odd}}\,\underbrace{\sin\frac{n\pi t}{2}}_{\text{odd}}\,dt \qquad \text{odd} \times \text{odd} = \text{even} \\
&= \int_{0}^{2} t\sin\frac{n\pi t}{2}\,dt \\
&= \left[\frac{-t\cos\frac{n\pi t}{2}}{n\pi/2}\right]_0^2 + \int_0^2 \frac{\cos\frac{n\pi t}{2}}{n\pi/2}\,dt \\
&= -\frac{4}{n\pi}\cos n\pi + \frac{2}{n\pi}\left[\frac{\sin\frac{n\pi t}{2}}{n\pi/2}\right]_0^2
\end{aligned}$$

so that

$$b_n = -\frac{4}{n\pi}(-1)^n$$

Recalling that $T_0 = 4$, the Fourier series is

$$\begin{aligned}
f(t) &= \sum_{n=1}^{\infty}\left(-\frac{4}{n\pi}(-1)^n\right)\sin\frac{n\pi t}{2} \\
&= -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi t}{2}
\end{aligned}$$

or, in expanded form,

$$f(t) = -\frac{4}{\pi}\left(-\sin\frac{\pi t}{2} + \tfrac{1}{2}\sin\pi t - \frac{1}{3}\sin\frac{3\pi t}{2} + \frac{1}{4}\sin 2\pi t + \cdots\right)$$

## 6.7 Example

Find the Fourier series of the function such that

$$f(t) = t^2 + t \qquad \text{for} \qquad -\pi < t < \pi$$

with $f(t + 2\pi) = f(t)$ for all $t$.

*Solution.* This function is **neither even nor odd**. It is $2\pi$-periodic so $T_0 = 2\pi$. Using this value for $T_0$ the formula for $a_n$ becomes

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} (t^2 + t)\cos nt\,dt \\
&= \frac{1}{\pi}\left[\left[(t^2 + t)\frac{\sin nt}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}(2t + 1)\frac{\sin nt}{n}\,dt\right] \qquad \text{if } n \neq 0
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{\pi n}\int_{-\pi}^{\pi}(2t+1)\sin nt\,dt \\
&= -\frac{1}{\pi n}\left[\left[-(2t+1)\frac{\cos nt}{n}\right]_{-\pi}^{\pi}+\int_{-\pi}^{\pi}\frac{2\cos nt}{n}\,dt\right] \\
&= -\frac{1}{\pi n}\left[-\frac{(2\pi+1)(-1)^n}{n}+\frac{(2(-\pi)+1)(-1)^n}{n}+\frac{2}{n}\left[\frac{\sin nt}{n}\right]_{-\pi}^{\pi}\right] \\
&= -\frac{1}{\pi n}\left[-\frac{4\pi(-1)^n}{n}\right]=\frac{4}{n^2}(-1)^n
\end{aligned}
$$

We have shown that

$$a_n = \frac{4}{n^2}(-1)^n \quad \text{if } n \neq 0$$

A separate calculation has to be done for $a_0$, since we obviously cannot put $n = 0$ into the above formula for $a_n$. Putting $n = 0$ into the original $a_n$ integral gives

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi}(t^2+t)\,dt = \frac{1}{\pi}\left[\frac{t^3}{3}+\frac{t^2}{2}\right]_{-\pi}^{\pi} \\
a_0 &= \frac{2\pi^2}{3}
\end{aligned}
$$

Next we find $b_n$. We have

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}(t^2+t)\sin nt\,dt \\
&= \frac{1}{\pi}\left[\left[-(t^2+t)\frac{\cos nt}{n}\right]_{-\pi}^{\pi}+\int_{-\pi}^{\pi}(2t+1)\frac{\cos nt}{n}\,dt\right] \\
&= \frac{1}{\pi}\left[-(\pi^2+\pi)\frac{(-1)^n}{n}+(\pi^2-\pi)\frac{(-1)^n}{n}+\frac{1}{n}\int_{-\pi}^{\pi}(2t+1)\cos nt\,dt\right] \\
&= \frac{1}{\pi}\left[-2\pi\frac{(-1)^n}{n}+\frac{1}{n}\left[\left[(2t+1)\frac{\sin nt}{n}\right]_{-\pi}^{\pi}-\int_{-\pi}^{\pi}\frac{2\sin nt}{n}\,dt\right]\right] \\
&= \frac{1}{\pi}\left[-2\pi\frac{(-1)^n}{n}-\frac{2}{n^2}\int_{-\pi}^{\pi}2\sin nt\,dt\right]
\end{aligned}
$$

so that

$$b_n = -\frac{2}{n}(-1)^n$$

Putting the formulae for $a_0$, $a_n$ and $b_n$ into the general Fourier series expansion, which when $T_0 = 2\pi$ is

$$f(t) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nt + b_n\sin nt)$$

gives

$$f(t) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}(-1)^n\cos nt - \frac{2}{n}(-1)^n\sin nt\right)$$

# 7 Complex form of a Fourier series

Recall that

$$j = \sqrt{-1}$$

Recall also, Euler's formula

$$e^{jt} = \cos t + j \sin t \tag{7.12}$$

and the other useful version of it:

$$e^{-jt} = \cos t - j \sin t \tag{7.13}$$

It is easy to see where (7.12) comes from. We simply expand $e^{jt}$ using Maclaurin series:

$$\begin{aligned} e^{jt} &= 1 + jt + \frac{(jt)^2}{2!} + \frac{(jt)^3}{3!} + \cdots \\ &= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots\right) + j\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right) \\ &= \cos t + j \sin t \end{aligned}$$

Formula (7.13) follows from (7.12) when we replace $t$ by $-t$.

Now, if we add equations (7.12) and (7.13) the $\sin t$ terms disappear and we get the following result:

$$\cos t = \frac{e^{jt} + e^{-jt}}{2} \tag{7.14}$$

Similarly, subtracting (7.13) from (7.12) gives

$$\sin t = \frac{e^{jt} - e^{-jt}}{2j} \tag{7.15}$$

Since $\cos t$ and $\sin t$ are periodic of period $2\pi$, so is $e^{jt}$. Furthermore, so are the functions $e^{njt}$ for any integer $n$ (positive or negative). This leads us to suppose that any $2\pi$ periodic function $f(t)$ can be represented as a sum of the functions $e^{njt}$ involving all integers $n$ (positive and negative).

With suitable adjustments to the period it can be shown that a similar statement can be made for a function $f(t)$ of period $T_0$. In fact

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{2jn\pi t/T_0} \tag{7.16}$$

where

$$c_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f(t) e^{-2jn\pi t/T_0} \, dt \qquad \text{for } n = 0, \pm 1, \pm 2, \pm 3, \ldots \tag{7.17}$$

Expression (7.16) with coefficients $c_n$ given by (7.17) is called the **complex form of the Fourier series**. Note that the sum is over **all** integer values of $n$ including negative ones.

The convergence properties of the complex form are the same as for the form we discussed earlier (expression (5.7)), i.e. the infinite sum in (7.16) converges to $f(t)$ unless there us a jump discontinuity in which case the sum converges to the midpoint of the jump (regardless of the actual value of $f(t)$ at such a point).

## 7.1 Example

Find the complex Fourier series of the function $f(t)$ such that

$$f(t) = \begin{cases} 0 & -\pi < t < -\pi/2 \\ 1 & -\pi/2 < t < \pi/2 \\ 0 & \pi/2 < t < \pi \end{cases}$$

with $f(t + 2\pi) = f(t)$ for all $t$.

*Solution.* The period is $2\pi$ so $T_0 = 2\pi$. With this value of $T_0$ expressions (7.16) and (7.17) reduce to

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{jnt}$$

where

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-jnt} \, dt \\ &= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{-jnt} \, dt = \frac{1}{2\pi} \left[ \frac{e^{-jnt}}{-jn} \right]_{-\pi/2}^{\pi/2} \\ &= \frac{1}{-2\pi jn} \left[ e^{-jn\pi/2} - e^{jn\pi/2} \right] \\ &= \frac{1}{\pi n} \left[ \frac{e^{jn\pi/2} - e^{-jn\pi/2}}{2j} \right] \end{aligned}$$

Recalling the formula $\sin x = \frac{e^{jx} - e^{-jx}}{2j}$, the above formula for $c_n$ can be put into the form

$$c_n = \frac{1}{\pi n} \sin \frac{n\pi}{2} \qquad \text{for } n = \pm 1, \pm 2, \pm 3, \ldots$$

The above calculation does not work if $n = 0$, because it has $n$ in the denominator, so we do a separate calculation for $n = 0$. The first line of the above calculation for $c_n$, with $n = 0$, gives

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \, dt = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} dt = \frac{1}{2}$$

So the complex Fourier series of the function is

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{jnt}$$

with the above expressions for $c_n$ and $c_0$.

It is possible to convert the complex form into a real form, as follows. We can write it in the form

$$f(t) = \tfrac{1}{2} + \sum_{n=-\infty}^{-1} c_n e^{jnt} + \sum_{n=1}^{\infty} c_n e^{jnt}$$

in which the $\frac{1}{2}$ at the front is the $n = 0$ term of the sum. Making the substitution $n = -m$ in the first sum of the above expression gives us

$$f(t) = \tfrac{1}{2} + \sum_{m=1}^{\infty} c_{-m} e^{-jmt} + \sum_{n=1}^{\infty} c_n e^{jnt}$$

and we can now simply replace $m$ by $n$ in the first sum, since they are dummy variables playing a similar role to the variable in a definite integral. This observation gives

$$\begin{aligned}
f(t) &= \tfrac{1}{2} + \sum_{n=1}^{\infty} c_{-n} e^{-jnt} + \sum_{n=1}^{\infty} c_n e^{jnt} \qquad \text{insert expression for } c_n \\
&= \tfrac{1}{2} + \sum_{n=1}^{\infty} \left\{ \frac{1}{-\pi n} \sin \frac{-n\pi}{2} e^{-jnt} + \frac{1}{\pi n} \sin \frac{n\pi}{2} e^{jnt} \right\} \\
&= \tfrac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{\pi n} \sin \frac{n\pi}{2} \underbrace{\left( e^{-jnt} + e^{jnt} \right)}_{=2\cos nt}
\end{aligned}$$

and so

$$f(t) = \tfrac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n} \sin \frac{n\pi}{2} \cos nt$$

We have converted the complex form into a real form. In the above sum the terms with $n = 2, 4, 6, \ldots$ are all zero. Replacing $n$ by $2n - 1$ has the effect of removing these zero terms to give

$$\begin{aligned}
f(t) &= \tfrac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi(2n-1)} \sin \frac{(2n-1)\pi}{2} \cos(2n-1)t \\
&= \tfrac{1}{2} + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{\pi(2n-1)} \cos(2n-1)t
\end{aligned}$$

which is more computationally efficient.

# 8 Amplitude and phase spectrum

Recall that the Fourier series of a periodic function of period $T_0$ (and frequency $\omega_0 = 2\pi/T_0$) is

$$f(t) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{2n\pi t}{T_0} + b_n \sin \frac{2n\pi t}{T_0} \right)$$

with certain formulae, namely (5.8) and (5.9), for the coefficients $a_n$ and $b_n$.

We can write the Fourier series in terms of $\omega_0$ as

$$\begin{aligned}
f(t) &= \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos n\omega_0 t + b_n \sin n\omega_0 t \right) \\
&= \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \delta_n)
\end{aligned}$$

where $A_n$ is called the amplitude and $\delta_n$ the phase. To find formulae for the numbers $A_n$ and $\delta_n$, $n = 1, 2, 3, \ldots$, we use elementary trigonometry. Now

$$A_n \cos(n\omega_0 t + \delta_n) = A_n \cos n\omega_0 t \cos \delta_n - A_n \sin n\omega_0 t \sin \delta_n$$

Comparing the right hand side of the above expression with

$$a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

we see that we need

$$A_n \cos \delta_n = a_n \qquad \text{and} \qquad A_n \sin \delta_n = -b_n$$

These expressions imply that

$$A_n = \sqrt{a_n^2 + b_n^2}$$

and

$$\tan \delta_n = -\frac{b_n}{a_n}$$

The numbers $\delta_n$, $n = 1, 2, 3, \ldots$ are called the **phase spectrum** and the numbers $A_n$, $n = 1, 2, 3, \ldots$ are called the **amplitude spectrum**. These two sets of numbers together form the spectrum and (with the frequency $\omega_0$) constitute one way of describing a periodic function.