Calculus (Spring) Sheet 2 solutions

1. $w_x = 2x/(x^2+y^2+z^2)$, $w_{xx} = \dfrac{-2x^2+2y^2+2z^2}{(x^2+y^2+z^2)^2}$.

Similarly $w_{yy} = \dfrac{-2y^2+2x^2+2z^2}{(x^2+y^2+z^2)^2}$ and $w_{zz} = \dfrac{-2z^2+2x^2+2y^2}{(x^2+y^2+z^2)^2}$. Therefore

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = \frac{2}{x^2+y^2+z^2}.$$

2. We know that

$$\delta S \approx \frac{\partial S}{\partial p}\,\delta p + \frac{\partial S}{\partial x}\,\delta x + \frac{\partial S}{\partial w}\,\delta w + \frac{\partial S}{\partial h}\,\delta h$$

which gives the stated result. The issue of sensitivity has to do with the coefficients of the small quantities δp, δx, δw and δh. With the given values, and after conversion to metres, the coefficient of δw is 1.13×10^8 while the coefficient of δh is 4.56×10^9. It is therefore more sensitive to changes in the quantity represented by h (i.e. thickness).

3. Need to maximise and minimise the directional derivative with respect to the unit vector $\mathbf{n}$. If θ is the angle between $\mathbf{n}$ and ∇f then

$$\begin{aligned} D_{\mathbf{n}} f(x,y) &= \nabla f \cdot \mathbf{n} = |\nabla f||\mathbf{n}|\cos\theta \\ &= |\nabla f|\cos\theta \quad \text{since } |\mathbf{n}| = 1. \end{aligned}$$

This is maximised when $\theta = 0$, i.e. when $\mathbf{n}$ is in the direction of ∇f, and it is minimised when $\theta = \pi$, i.e. when $\mathbf{n}$ is in the direction of $-\nabla f$.
Along the path of steepest descent, (dx, dy) (which represents your direction at a particular point on the curve) would have to be parallel to ∇z and thus, for some λ,

$$(dx, dy) = \lambda \nabla z = \lambda \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y} \right)$$

Therefore $dx = \lambda \partial z/\partial x = 2x\lambda$ and $dy = \lambda \partial z/\partial y = 6y\lambda$ and so

$$\frac{dy}{dx} = \frac{3y}{x}$$

Also we start at the point $(1,1,4)$ on the surface, so that $y = 1$ when $x = 1$. Solving the above differential equation subject to this condition gives $y = x^3$.

5. When you solve $f_x = 0$ and $f_y = 0$ for x and y you will be solving

$$y^2 - 2x + 2y = 0, \qquad 2xy - 6y^2 + 2x = 0.$$

The first of these equations implies $y^2 + 2y = 2x$, and then the second one becomes $(y^2+2y)y - 6y^2 + y^2 + 2y = 0$.
Answers: $(0,0)$ saddle, $(\frac{3}{2}, 1)$ maximum, $(4,2)$ saddle.

7. Minimising cost is not the same as minimising area, since different parts of the crate are made of different materials. If you minimise area and then convert to cost at the end your answer will be wrong. So you must work with cost from the outset. The cost C in pence is given by $C = 30xy + 20xz + 20yz$. The volume is 96 so $xyz = 96$ and therefore the cost can be expressed in terms of x and y as

$$C(x, y) = 30xy + \frac{1920}{y} + \frac{1920}{x}.$$

From this point on it is similar to an example done in lectures. The answer is £ 14.40.