Calculus (Spring) Sheet 4 solutions

1. (i) Substitute $u = x - 2y$ and $v = 3x - y$. The Jacobian of the transformation is $\frac{1}{5}$ and so

$$\iint_D \left(\frac{x-2y}{3x-y}\right) dA = \int_1^8 \int_0^4 \frac{u}{5v}\, du\, dv = \int_1^8 \left[\frac{u^2}{10v}\right]_{u=0}^{u=4} dv = \frac{8}{5}\int_1^8 \frac{dv}{v} = \frac{8}{5}\ln 8$$

(ii) Transform into plane polars. The Jacobian is r, and the integral becomes

$$\begin{aligned} \iint_D \sqrt{4-x^2-y^2}\, dA &= \int_0^{2\pi}\int_0^2 \sqrt{4-r^2}\, r\, dr\, d\theta \\ &= \int_0^{2\pi} \left[-\frac{1}{3}(4-r^2)^{3/2}\right]_{r=0}^{r=2} d\theta \\ &= \int_0^{2\pi} \frac{8}{3}\, d\theta = \frac{16\pi}{3} \end{aligned}$$

2.

$$\begin{aligned} \int_0^\infty \frac{\tan^{-1}\pi x - \tan^{-1} x}{x}\, dx &= \int_0^\infty \frac{1}{x}\left(\int_x^{\pi x} \frac{dy}{1+y^2}\right) dx \\ &= \int_0^\infty \int_x^{\pi x} \frac{dy\, dx}{x(1+y^2)} \end{aligned}$$

The region of integration is therefore the wedge $\{(x,y) :\ x \le y \le \pi x,\ 0 \le x < \infty\}$. Changing the order of integration from $dy\, dx$ to $dx\, dy$ gives

$$\begin{aligned} \int_0^\infty \frac{\tan^{-1}\pi x - \tan^{-1} x}{x}\, dx &= \int_0^\infty \int_{y/\pi}^{y} \frac{dx\, dy}{x(1+y^2)} = \int_0^\infty \left[\frac{1}{1+y^2}\ln x\right]_{x=y/\pi}^{x=y} dy \\ &= \int_0^\infty \frac{1}{1+y^2}(\ln y - \ln(y/\pi))\, dy \\ &= \ln\pi \int_0^\infty \frac{1}{1+y^2}\, dy = \ln\pi \left[\tan^{-1} y\right]_{y=0}^{y=\infty} = \frac{\pi}{2}\ln\pi \end{aligned}$$

3.

$$\begin{aligned} \text{integral} &= \int_0^2 \int_0^{z^2} \left[ze^x\right]_{x=0}^{x=\ln y} dy\, dz \\ &= \int_0^2 \int_0^{z^2} z(y-1)\, dy\, dz = \int_0^2 \left[z\left(\frac{y^2}{2} - y\right)\right]_{y=0}^{y=z^2} dz \\ &= \int_0^2 \left(\frac{z^5}{2} - z^3\right) dz = \frac{4}{3}. \end{aligned}$$

4. Convert to cylindrical polars. Then

$$\iiint_V \frac{1}{x^2+y^2}\, dx\, dy\, dz = \int_0^1 \int_0^{2\pi} \int_1^3 \frac{1}{r^2} \underbrace{r}_{\text{Jacobian}} dr\, d\theta\, dz = 2\pi \ln 3.$$

5. In terms of spherical polars, $z = \sqrt{x^2 + y^2}$ becomes $r\cos\theta = \sqrt{r^2\sin^2\theta} = r\sin\theta$ so that $\tan\theta = 1$ and $\theta = \pi/4$. The volume of any object is $\iiint_V dx\,dy\,dz$ where V is the object. Converting into spherical polars and inserting the Jacobian for the transformation which is $r^2\sin\theta$, gives

$$\text{volume} = \int_0^{2\pi}\int_0^{\pi/4}\int_0^4 r^2\sin\theta\,dr\,d\theta\,d\phi = \frac{64\pi}{3}(2-\sqrt{2}).$$

6. Substitute

$$\begin{aligned} u &= \mathbf{a}\cdot\mathbf{r} = a_1x + a_2y + a_3z, \\ v &= \mathbf{b}\cdot\mathbf{r} = b_1x + b_2y + b_3z, \\ w &= \mathbf{c}\cdot\mathbf{r} = c_1x + c_2y + c_3z. \end{aligned}$$

Then we can say that

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where A is the matrix with rows $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. The determinant of A is given by $\det A = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ by a result about the triple scalar product (Autumn Semester Linear Algebra). However, we need the Jacobian of the transformation from (x, y, z) to (u, v, w); the relationship being

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

and by standard theory of determinants,

$$\det A^{-1} = (\det A)^{-1} = \frac{1}{\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})}.$$

Recall that it is the *modulus* of the Jacobian that we put into the integral. Thus,

$$\begin{aligned} \iiint_V (\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})(\mathbf{c}\cdot\mathbf{r})\,dV &= \int_0^\alpha\int_0^\beta\int_0^\gamma uvw \underbrace{\left(\frac{1}{|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|}\right)}_{\text{Jacobian}} du\,dv\,dw \\ &= \frac{1}{|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|}\int_0^\alpha\int_0^\beta\int_0^\gamma uvw\,du\,dv\,dw \\ &= \frac{(\alpha\beta\gamma)^2}{8|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|}. \end{aligned}$$

7. By Green's theorem, letting A be the rectangular region,

$$\begin{aligned} \int_C (x^2y^3 - 3y)\,dx + x^3y^2\,dy &= \iint_A (3x^2y^2 - (3x^2y^2 - 3))\,dA \\ &= 3\iint_A dA \\ &= 3(\text{area}) \end{aligned}$$

8. By Green's theorem

$$
\begin{aligned}
\text{integral} &= \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA \\
&= \iint ((2y + 10x) - (3x + 2y))\, dA \\
&= \iint 7x\, dA \qquad\qquad \text{now convert into polars} \\
&= \int_0^{2\pi} \int_0^1 7(1 + r\cos\theta) \underbrace{r}_{\text{Jacobian}} dr\, d\theta \\
&= 7\pi.
\end{aligned}
$$