Faculty of Engineering and Physical Sciences

Mathematics 1b: tutorial sheet 8

Exercise 1:

Evaluate A+B, A-B, 4A and 4A+B for each of the following cases:
When adding or subtracting matrices you should add or subtract term by term. In scalar multiplication (i.e. when you multiply a matrix by a number) you just multiply each element of the matrix by that number. However, when you multiply a matrix by another matrix it is more complicated. For multiplying matrices the lecturer showed you one method and I will show you another one in my solution to Exercise 2 below.

(i)

$$A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 4 & 0 & 2 & 1 \\ 2 & -5 & 1 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 3 & -4 & 1 & 2 \\ 1 & 5 & 0 & 3 \\ 2 & -2 & 3 & -1 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 1+3 & 2-4 & -1+1 & 0+2 \\ 4+1 & 0+5 & 2+0 & 1+3 \\ 2+2 & -5-2 & 1+3 & 2-1 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 0 & 2 \\ 5 & 5 & 2 & 4 \\ 4 & -7 & 4 & 1 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 1-3 & 2+4 & -1-1 & 0-2 \\ 4-1 & 0-5 & 2-0 & 1-3 \\ 2-2 & -5+2 & 1-3 & 2+1 \end{pmatrix} = \begin{pmatrix} -2 & 6 & -2 & -2 \\ 3 & -5 & 2 & -2 \\ 0 & -3 & -2 & 3 \end{pmatrix}$$

$$4A = 4\begin{pmatrix} 1 & 2 & -1 & 0 \\ 4 & 0 & 2 & 1 \\ 2 & -5 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 4\times 1 & 4\times 2 & 4\times(-1) & 4\times 0 \\ 4\times 4 & 4\times 0 & 4\times 2 & 4\times 1 \\ 4\times 2 & 4\times(-5) & 4\times 1 & 4\times 2 \end{pmatrix} = \begin{pmatrix} 4 & 8 & -4 & 0 \\ 16 & 0 & 8 & 4 \\ 8 & -20 & 4 & 8 \end{pmatrix}$$

$$4A + B = \begin{pmatrix} 4 & 8 & -4 & 0 \\ 16 & 0 & 8 & 4 \\ 8 & -20 & 4 & 8 \end{pmatrix} + \begin{pmatrix} 3 & -4 & 1 & 2 \\ 1 & 5 & 0 & 3 \\ 2 & -2 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 4+3 & 8-4 & -4+1 & 0+2 \\ 16+1 & 0+5 & 8+0 & 4+3 \\ 8+2 & -20-2 & 4+3 & 8-1 \end{pmatrix}$$

$$4A + B = \begin{pmatrix} 7 & 4 & -3 & 2 \\ 17 & 5 & 8 & 7 \\ 10 & -22 & 7 & 7 \end{pmatrix}$$

(ii)

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$$

$$B = \begin{pmatrix} -3 & -2 \\ 1 & -5 \\ 4 & 3 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 1-3 & 2-2 \\ 3+1 & 4-5 \\ 5+4 & 6+3 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 4 & -1 \\ 9 & 9 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 1+3 & 2+2 \\ 3-1 & 4+5 \\ 5-4 & 6-3 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 2 & 9 \\ 1 & 3 \end{pmatrix}$$

$$4A = 4\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 4\times 1 & 4\times 2 \\ 4\times 3 & 4\times 4 \\ 4\times 5 & 4\times 6 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 12 & 16 \\ 20 & 24 \end{pmatrix}$$

$$4A + B = \begin{pmatrix} 4 & 8 \\ 12 & 16 \\ 20 & 24 \end{pmatrix} + \begin{pmatrix} -3 & -2 \\ 1 & -5 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 4-3 & 8-2 \\ 12+1 & 16-5 \\ 20+4 & 24+3 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 13 & 11 \\ 24 & 27 \end{pmatrix}$$

Exercise 2:
Find where possible the matrix products AB and BA in each of the following cases:

(i)

$$A = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$$

$$B = \begin{pmatrix} 6 & 8 \\ 1 & -4 \end{pmatrix}$$

$$A \times B = \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix} \times \begin{pmatrix} 6 & 8 \\ 1 & -4 \end{pmatrix} = \begin{pmatrix} 2\times 6 + 4\times 1 & 2\times 8 + 4\times(-4) \\ 3\times 6 - 1\times 1 & 3\times 8 - 1\times(-4) \end{pmatrix} = \begin{pmatrix} 16 & 0 \\ 17 & 28 \end{pmatrix}$$

$$B \times A = \begin{pmatrix} 6 & 8 \\ 1 & -4 \end{pmatrix} \times \begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 6\times 2 + 8\times 3 & 6\times 4 + 8\times(-1) \\ 1\times 2 + (-4)\times 3 & 1\times 4 - 4\times(-1) \end{pmatrix} = \begin{pmatrix} 36 & 16 \\ -10 & 8 \end{pmatrix}$$

There is a small trick that may help you out to multiply matrices, it only consists into writing the matrices in a slightly different manner on your page and I find it very helpful. If you find it confusing just refer to the method you have been taught in the lectures.

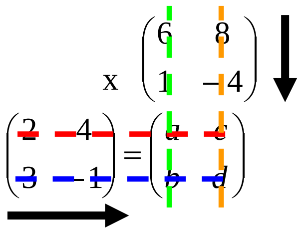
Here goes:

First write the matrices as follows (I am using the previous example for this), so that the matrix you multiply A by, (B) is slightly shifted towards the right hand side and higher up:

$$\qquad\qquad\begin{pmatrix} 6 & 8 \\ 1 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$$

Then draw some lines as below:



The result of the multiplication will be a four by four matrix as well $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$. a corresponds to the intersection of the red and green line and therefore will be the result of the multiplication of numbers which are on the first row of matrix A and first column of matrix B. Besides, to choose your numbers you always go from left to right and top to bottom. Hence a = 2x6 + 4x1 = 16. Similarly b corresponds to the intersection of the blue and green lines and is a result of multiplication on numbers on the second row of A and first column of B: b = 3x6 + (-1)x1 = 17. It then follows that c corresponds to the intersection of red and yellow lines and that d corresponds to the intersection of the blue and yellow line. This method is always valid as long as it is possible to multiply the matrices you are considering. It is worthy to note at this stage that it is only possible to multiply matrices if the number of columns of the first matrix equals the number of rows of the second.

(ii)

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{pmatrix}$$

$$A \times B = \begin{pmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1-2+1 & 2-4+2 & 3-6+3 \\ -3+4-1 & -6+8-2 & -9+12-3 \\ -2+2+0 & -4+4+0 & -6+6+0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$B \times A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{pmatrix} \times \begin{pmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1-6-6 & -1+4+3 & 1-2+0 \\ 2-12-12 & -2+8+6 & 2-4+0 \\ 1-6-6 & -1+4+3 & 1-2+0 \end{pmatrix} = \begin{pmatrix} -11 & 6 & -1 \\ -22 & 12 & -2 \\ -11 & 6 & -1 \end{pmatrix}$$

(iii)

$$A = \begin{pmatrix} -4 & 6 & 2 \\ -2 & -2 & 3 \\ 1 & 1 & 8 \end{pmatrix}$$

$$B = \begin{pmatrix} -2 & 4 & 7 & 6 \\ -3 & 1 & 2 & 4 \\ 0 & 1 & 6 & -2 \end{pmatrix}$$

For this example, it is only possible to multiply A by B and not the other way round. This is because A has three columns and B three rows but B has four columns and A three rows.

$$A \times B = \begin{pmatrix} -4 & 6 & 2 \\ -2 & -2 & 3 \\ 1 & 1 & 8 \end{pmatrix} \times \begin{pmatrix} -2 & 4 & 7 & 6 \\ -3 & 1 & 2 & 4 \\ 0 & 1 & 6 & -2 \end{pmatrix} = \begin{pmatrix} 8-18+0 & -16+6+2 & -28+12+12 & -24+24-4 \\ 4+6+0 & -8-2+3 & -14-4+18 & -12-8-6 \\ -2-3+0 & 4+1+8 & 7+2+48 & 6+4-16 \end{pmatrix}$$

$$A \times B = \begin{pmatrix} -10 & -8 & -4 & -4 \\ 10 & -7 & 0 & -26 \\ -5 & 13 & 57 & -6 \end{pmatrix}$$

Exercise 3:

Find AB if:

$A = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$

$B = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$

$$A \times B = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix} \times \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = 4 \times 2 + 5 \times 3 + 6 \times (-1) = 17$$

Exercise 4:

Find A^2 and A^3 when:

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4+0+1 & -2-1 & 1 \\ 2 & 1 & 4 \\ 2+1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & -3 & 1 \\ 2 & 1 & 4 \\ 3 & -1 & 2 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 5 & -3 & 1 \\ 2 & 1 & 4 \\ 3 & -1 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 10+0+1 & -5-3+0 & 5-6+1 \\ 4+0+4 & -2+1+0 & 2+2+4 \\ 6-0+2 & -3-1+0 & 3-2+2 \end{pmatrix} = \begin{pmatrix} 11 & -8 & 0 \\ 8 & -1 & 8 \\ 8 & -4 & 3 \end{pmatrix}$$

A^3 may be calculated either as A x A^2 or A^2 x A. Therefore:

$$A^3 = A \times A^2 = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 5 & -3 & 1 \\ 2 & 1 & 4 \\ 3 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 10-2+3 & -6-1-1 & 2-4+2 \\ 0+2+6 & 0+1-2 & 0+4+4 \\ 5+0+3 & -3+0-1 & 1+0+2 \end{pmatrix} = \begin{pmatrix} 11 & -8 & 0 \\ 8 & -1 & 8 \\ 8 & -4 & 3 \end{pmatrix}$$

Exercise 5:

Find the ranks of the following matrices by reducing them to row echelon form. Once you have done so using the Gauss Jordan elimination, the rank of the matrix is equal to the number of rows which contain at least one non nil number as long as you cannot perform any further elimination.

(i)

$$\begin{pmatrix} 1 & 2 & 4 \\ -2 & 0 & 1 \\ -1 & 2 & 5 \end{pmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 4 & 9 \\ -1 & 2 & 5 \end{pmatrix} \xrightarrow{R_3 \to R_3 + R_1} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 4 & 9 \\ 0 & 4 & 9 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 4 & 9 \\ 0 & 0 & 0 \end{pmatrix}$$

The rank of this matrix is therefore equal to 2.

(ii)

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 1 & 3 & 4 & 5 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{pmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

As above the rank of the matrix is 2.

(iii)

$$\begin{pmatrix} 2 & 4 & 3 & 4 \\ 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 3 & 7 & 4 & 6 \end{pmatrix} \xrightarrow{R_1 \to R_3} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{pmatrix} \xrightarrow{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 3R_1}} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \xrightarrow{R_4 \to R_4 - R_2} \begin{pmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

To solve this one you have to substitute row one for row three. Then you obtain three rows that do not contain a full line of zeros, therefore the rank is equal to 3.

(iv)

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{pmatrix} \xrightarrow{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 6R_1}} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{pmatrix} \xrightarrow{\substack{R_2 \to R_2 + R_3 \\ R_3 \to R_3 - R_4}} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & 3 & -2 \\ 0 & -4 & -11 & 5 \end{pmatrix} \xrightarrow{R_4 \to R_4 - R_2} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & -4 & -11 & 5 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Once again the rank is equal to 3.

(v)

$$\begin{pmatrix} 0 & -1 & 2 \\ 2 & 0 & 2 \\ 3 & 4 & 7 \end{pmatrix} \xrightarrow{\substack{R_1 \to R_1 + R_2 \\ R_2 \to 3R_2 - 2R_3}} \begin{pmatrix} 2 & -1 & 4 \\ 0 & -8 & -8 \\ 3 & 4 & 7 \end{pmatrix} \xrightarrow{R_3 \to 2R_3 - 3R_1} \begin{pmatrix} 2 & -1 & 4 \\ 0 & -8 & -8 \\ 0 & 11 & 2 \end{pmatrix}$$

No further full elimination of a row is possible; therefore this matrix is of rank 3.

Exercise 6:

Solve the following system of equations by reducing them to row echelon form. The first thing you have to do is to write the augmented matrix, then to proceed to elimination as before while not forgetting you also transform the right hand side of the augmented matrix as you do so.

$$\begin{aligned} x_1 - 5x_2 + 3x_3 &= -1 \\ 4x_1 - 18x_2 + 15x_3 &= 4 \\ -2x_1 + 11x_2 - 11x_3 &= -7 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & -5 & 3 & -1 \\ 4 & -18 & 15 & 4 \\ -2 & 11 & -11 & -7 \end{array}\right) \xrightarrow{\substack{R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 + 2R_1}} \left(\begin{array}{ccc|c} 1 & -5 & 3 & -1 \\ 0 & 2 & 3 & 8 \\ 0 & 1 & -5 & -9 \end{array}\right) \xrightarrow{R_3 \to 2R_3 - R_2} \left(\begin{array}{ccc|c} 1 & -5 & 3 & -1 \\ 0 & 2 & 3 & 8 \\ 0 & 0 & -13 & -26 \end{array}\right)$$

$$\begin{aligned} &R_3 \Rightarrow -13x_3 = -26 \Rightarrow x_3 = 2 \\ &R_2 \Rightarrow 2x_2 + 3x_3 = 8 = 2x_2 + 6 \Rightarrow 2x_2 = 2 \Rightarrow x_2 = 1 \\ &R_1 \Rightarrow x_1 - 5x_2 + 3x_3 = -1 = x_1 - 5 + 6 = x_1 + 1 \Rightarrow x_1 = -2 \end{aligned}$$

$$\begin{aligned} x_1 &= -2 \\ x_2 &= 1 \\ x_3 &= 2 \end{aligned}$$

Exercise 7:

Find the value of λ for which the equations (see below) are consistent, and solve them for this value of λ.

$$\begin{aligned} x_1 - 2x_2 + 9x_3 &= 4 \\ 2x_1 - 6x_2 + 21x_3 &= 9 \\ 3x_1 + 10x_3 &= 6 \\ -x_1 + 4x_2 - 13x_3 &= \lambda \end{aligned}$$

First write the augmented matrix. Then you have to process to the same elimination as for the previous exercise.

$$\left(\begin{array}{ccc|c} 1 & -2 & 9 & 4 \\ 2 & -6 & 21 & 9 \\ 3 & 0 & 10 & 6 \\ -1 & 4 & -13 & \lambda \end{array}\right) \xrightarrow[R_4\to R_4+R_1]{\substack{R_2\to R_2-2R_1 \\ R_3\to R_3-3R_1}} \left(\begin{array}{ccc|c} 1 & -2 & 9 & 4 \\ 0 & -2 & 3 & 1 \\ 0 & 6 & -17 & -6 \\ 0 & 2 & -4 & 4+\lambda \end{array}\right) \xrightarrow{\substack{R_3\to R_3+3R_2 \\ R_4\to R_4+R_2}} \left(\begin{array}{ccc|c} 1 & -2 & 9 & 4 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & -8 & -3 \\ 0 & 0 & -1 & 5+\lambda \end{array}\right)$$

$$\xrightarrow{R_4\to 8R_4-R_3} \left(\begin{array}{ccc|c} 1 & -2 & 9 & 4 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & -8 & -3 \\ 0 & 0 & 0 & 8(5+\lambda)+3 \end{array}\right)$$

$$R_4 \Rightarrow 0 = 8(5+\lambda)+3 \Rightarrow 8\lambda = -3-40 = -43 \Rightarrow \lambda = -\frac{43}{8}$$

$$R_3 \Rightarrow -8x_3 = -3 \Rightarrow x_3 = \frac{3}{8}$$

$$R_2 \Rightarrow -2x_2 + 3x_3 = 1 = -2x_2 + \frac{9}{8} \Rightarrow x_2 = \frac{1-\frac{9}{8}}{-2} = \frac{\frac{-1}{8}}{-2} = \frac{1}{16}$$

$$R_1 \Rightarrow x_1 - 2x_2 + 9x_3 = 4 = x_1 - 2\times\frac{1}{16} + 9\times\frac{3}{8} \Rightarrow x_1 = \frac{32+1-27}{8} = \frac{6}{8} = \frac{3}{4}$$

$$x_1 = \frac{3}{4}$$

$$x_2 = \frac{1}{16}$$

$$x_3 = \frac{3}{8}$$

$$\lambda = -\frac{43}{8}$$

MLA 17/04/07