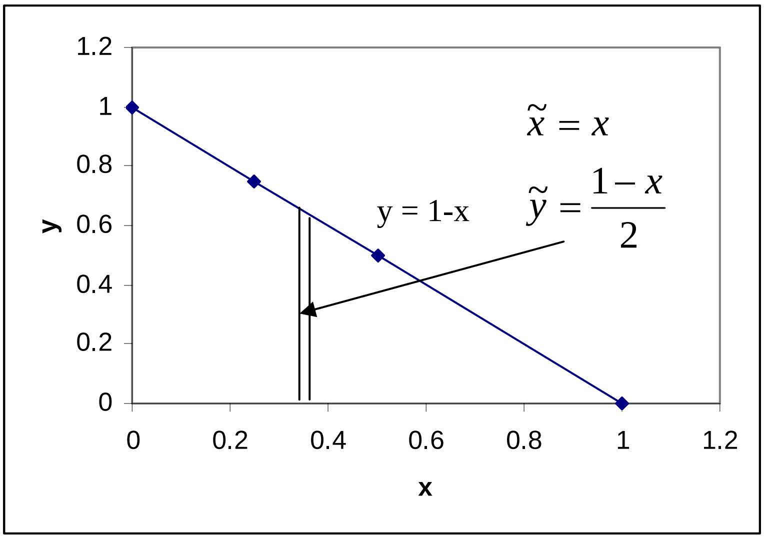
Faculty of Engineering and Physical Sciences

Mathematics 1b: tutorial sheet 3

Exercise 1:
Find the position of the centre of mass of:

(i) the triangular lamina formed by y = 1-x and the x and y axes.

What you always have to do first is to read properly the text of your exercise. Here even before you start, you are told that the lamina is a triangle. Then it is a good idea to sketch it so that what you are doing is clearer in your mind.



To obtain the centre of mass you have first to define the coordinates of the centre of the strip. This strip is infinitesimally thin and it is assumed that each side is of the same height. The coordinates are written as $(\tilde{x}, \tilde{y})$.

$$\tilde{x} = x$$

$$\tilde{y} = \frac{1-x}{2}$$

Some of you seem to have trouble defining those coordinates but if you observe the centre of the strip, it becomes obvious that the value of y is defined as being half of the maximum height of the strip which is 1-x. Another difficulty consisted in plotting the straight line; remember that if the coefficient in x is negative the line will be a decreasing function as above.

The coordinates of the centre of mass are:

$$\bar{x} = \frac{\int \tilde{x} dm}{\int dm}$$

$$\bar{y} = \frac{\int \tilde{y} dm}{\int dm}$$

Where dm is the mass of your infinitesimally thin vertical strip cutting across the lamina, and $(\tilde{x}, \tilde{y})$ the centre of this strip. dm has to be defined using the mass per unit area of the strip, σ. To do this you first define the area you are considering and then multiply it by σ. Basically your strip is a rectangle with width dx and height y or 1-x. The area will be (1-x)dx and dm will be defined as follows:

$$dm = \sigma(1-x)dx$$

$$\bar{x} = \frac{\int \tilde{x} dm}{\int dm} = \frac{\int_0^1 \sigma x(1-x)dx}{\int_0^1 \sigma(1-x)dx} = \frac{[\frac{x^2}{2} - \frac{x^3}{3}]_0^1}{[x - \frac{x^2}{2}]_0^1} = \frac{\frac{1}{2} - \frac{1}{3}}{1 - \frac{1}{2}} = \frac{\frac{3-2}{6}}{\frac{1}{2}} = \frac{1}{3}$$

$$\bar{y} = \frac{\int \tilde{y} dm}{\int dm} = \frac{\int_0^1 \sigma(\frac{1-x}{2})(1-x)dx}{\int_0^1 \sigma(1-x)dx} = \frac{1}{2}\frac{\int_0^1 (1-2x+x^2)dx}{\int_0^1 (1-x)dx} = \frac{1}{2}\frac{[x - x^2 + \frac{x^3}{3}]_0^1}{\frac{1}{2}} = 1 - 1 + \frac{1}{3} = \frac{1}{3}$$

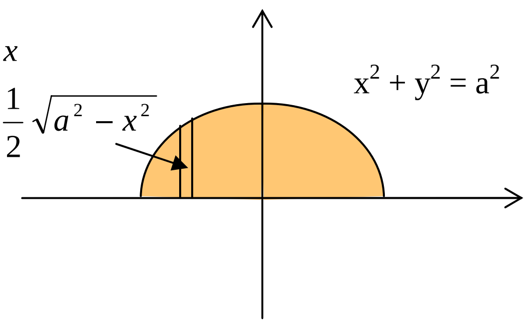
$$(\bar{x}, \bar{y}) = (\frac{1}{3}, \frac{1}{3})$$

(ii)

As previously start with a graph.

$$\tilde{x} = x$$

$$\tilde{y} = \frac{1}{2}\sqrt{a^2 - x^2}$$



The area of the lamina is coloured in orange. The coordinates of the centre of the strip are defined as:

$$\tilde{x} = x$$

$$\tilde{y} = \frac{1}{2}\sqrt{a^2 - x^2}$$

$$dm = \sigma\sqrt{a^2 - x^2}\,dx$$

$$\bar{x} = 0$$

$$\bar{y} = \frac{\int_{-a}^{a} \frac{1}{2}\sqrt{a^2 - x^2} \times \sigma\sqrt{a^2 - x^2}\,dx}{\int_{-a}^{a} \sigma\sqrt{a^2 - x^2}\,dx} = \frac{1}{2}\frac{\int_{-a}^{a} (a^2 - x^2)dx}{\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx}$$

$\bar{x} = 0$ by symmetry, considering the shape of the laminate, the centre of mass will be situated on the y axis, hence $\bar{x} = 0$. Let calculate the y coordinate of the centre of mass in two parts, according to which integral we are looking at:

$$\int_{-a}^{a} (a^2 - x^2)dx = [a^2 x - \frac{x^3}{3}]_{-a}^{a} = a^3 - \frac{a^3}{3} + a^3 - \frac{a^3}{3} = 2a^3 - \frac{2a^3}{3} = \frac{6a^3 - 2a^3}{3} = \frac{4a^3}{3}$$

$$\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx$$

$$x = a\sin\theta \Rightarrow dx = a\cos\theta d\theta$$

$$x^2 = a^2\sin^2\theta \Rightarrow a^2 - x^2 = a^2(1 - \sin^2\theta) = a^2\cos^2\theta \Rightarrow \sqrt{a^2 - x^2} = a\cos\theta$$

$Limits$:

$$x = a\sin\theta$$

$$x = a \Rightarrow \sin\theta = \pm 1 \Rightarrow \theta = \pm\frac{\pi}{2}$$

$$\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx = \int_{-\pi/2}^{\pi/2} a\cos\theta \times a\cos\theta d\theta = \int_{-\pi/2}^{\pi/2} a^2\cos^2\theta d\theta$$

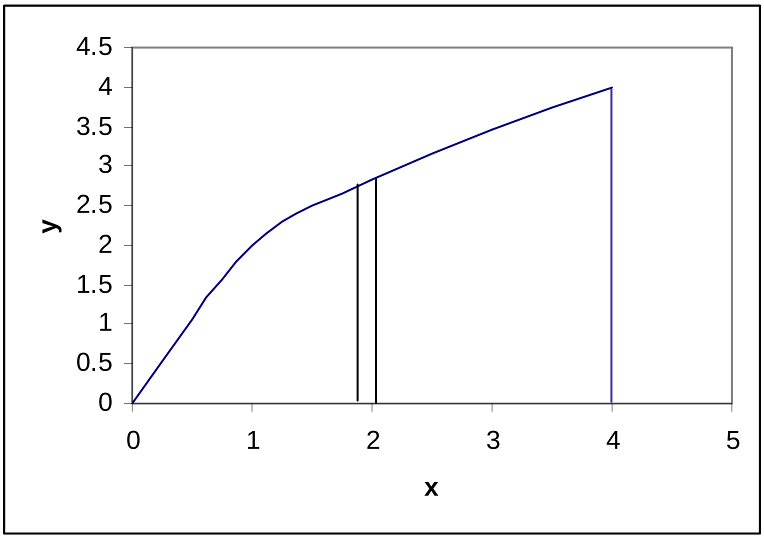
$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \Rightarrow$$

$$\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx = \int_{-\pi/2}^{\pi/2} a^2\cos^2\theta d\theta = 2\int_{0}^{\pi/2} a^2\cos^2\theta d\theta = 2\int_{0}^{\pi/2} a^2(\frac{1 + \cos 2\theta}{2})d\theta = a^2[\theta + \frac{1}{2}\sin 2\theta]_0^{\pi/2} = \frac{\pi}{2}a^2$$

$$\Rightarrow$$

$$\bar{y} = \frac{1}{2} \times \frac{\frac{4a^3}{3}}{\frac{\pi}{2}a^2} = \frac{2a3}{3} \times \frac{2}{\pi} = \frac{4a}{3\pi} \Rightarrow (\bar{x}, \bar{y}) = (0, \frac{4a}{3\pi})$$

Exercise 2:
A lamina is bounded by the curve $y^2 = 4x$, the x axis and the line x = 4. A sketch of this lamina is as follows:



For a rod rotating at one end, the moment of inertia is (as calculated during the lecture):

$$I = \frac{1}{3}ma^2$$

Mass of rod in diagram = σ × area= σydx

$$I_{rod} = \frac{1}{3}(\sigma y dx)y^2 \Rightarrow I_{la\min a} = \int_0^4 \frac{1}{3}\sigma y^3 dx$$

$$y^2 = 4x \Rightarrow y = 2x^{1/2} \Rightarrow y^3 = 8x^{3/2}$$

$$I_{la\min a} = \int_0^4 8\times\frac{1}{3}\sigma x^{3/2}dx = \frac{8\sigma}{3}[\frac{x^{(\frac{3}{2}+1)}}{\frac{3}{2}+1}]_0^4 = \frac{8\sigma}{3}[\frac{2}{5}x^{5/2}]_0^4 = \frac{8\sigma}{3}\times[\frac{64}{5}] = \frac{512\sigma}{15}$$

To complete this exercise, you have to express the moment of inertia as a function of the mass of the lamina. To do so, you can calculate the mass as a function of σ, then express σ as a function of mass and replace it in the expression of the moment of inertia for the lamina:

$$m = \sigma(area) = \sigma\int_0^4 ydx = \sigma\int_0^4 2x^{1/2}dx = 2\sigma\times[\frac{2}{3}x^{3/2}]_0^4 = 2\sigma\times\frac{2}{3}\times 8 = \frac{32\sigma}{3}$$

$$m = \frac{32\sigma}{3} \Rightarrow \sigma = \frac{3m}{32}$$

$$I_{la\min a} = \frac{512\sigma}{15} = \frac{512}{15}\times\frac{3m}{32} = \frac{16m}{5}$$

Exercise 3:

Evaluate the following double integrals:

(i)

$$\int_0^1\int_{-1}^1 (x+y+1)dxdy = \int_0^1 [\frac{x^2}{2}+xy+x]_{-1}^1 dy = \int_0^1 (\frac{1}{2}+y+1-\frac{1}{2}+y+1)dy$$

$$\int_0^1\int_{-1}^1 (x+y+1)dxdy = \int_0^1 (2y+2)dy = [y^2+2y]_0^1 = (1+2-0) = 3$$

To calculate double integrals, always look at which variable is written first as dx or dy. Here it is dx, meaning that you have to integrate in x first. While you are integrating in x, y behaves as a constant and vice versa.

(ii)

$$\int_0^{Ln3}\int_0^{Ln2} e^{x+y}dxdy = \int_0^{Ln3} e^y[e^x]_0^{Ln2}dy = \int_0^{Ln3} e^y(e^{Ln2}-e^0)dy = \int_0^{Ln3} e^y(2-1)dy = [e^y]_0^{Ln3} = (e^{Ln3}-e^0) = 3-1 = 2$$

(iii)

$$\int_0^2\int_0^{9-4x^2} 3xdydx = \int_0^2 [3xy]_0^{9-4x^2}dx = \int_0^2 [3x\times(9-4x^2)]dx = \int_0^2 (27x-12x^3)dx = [27\frac{x^2}{2}-3x^4]_0^2$$

$$\int_0^2\int_0^{9-4x^2} 3xdydx = 27\times 2-3\times 16 = 6$$

MLA 21/03/07