Answers to Mathematics 1b exam papers

Spring 2002

1. (i) $y = \frac{1}{x^2+d}$, (ii) $y = \tan(\frac{x^3}{3} + \frac{\pi}{4})$, (iii) $y = \frac{x}{3} - \frac{1}{9} + c\,e^{-3x}$
2. 10.44 am
3. (i) $y = Ae^{-2x} + Be^{-5x}$, (ii) $y = e^{-x}(A\cos 2x + B\sin 2x) + \frac{1}{5}x^2 - \frac{4}{25}x - \frac{27}{125}$
4. (i) $\dfrac{\partial z}{\partial x} = 3x^2 - y^2$, $\dfrac{\partial z}{\partial y} = -2xy + 1$,
(ii) $\dfrac{\partial z}{\partial x} = e^{x/y}$, $\dfrac{\partial z}{\partial y} = -\frac{x}{y}e^{x/y} + e^{x/y}$
5. $1 \times 1 \times 2$
6. (i) $(x, y, z) = (\frac{2}{3}, \frac{1}{3}, -\frac{1}{3})$
(ii) no solution (because $\text{Rank}(A) = 2$ and $\text{Rank}(A|\mathbf{b}) = 3$)
7. (i) 3, (ii) -2
8. The cone can be made up of infinitely many thin disks stacked alongside each other. A typical thin disk is perpendicular to the x-axis (with the axis running through its centre), one face of the disk being at point x and the other at $x + \delta x$. The moment of inertia of the disk is "$\frac{1}{2}Ma^2$". Here M is the mass of the disk which equals density times volume, i.e. $\rho\pi y^2\delta x$. But the disk's radius y is $2x$, so the disks mass is $4\rho\pi x^2\delta x$. The disk's moment of inertia δI is therefore $\delta I = 2\rho\pi x^2\delta x(2x)^2 = 8\rho\pi x^4\delta x$. The moment of inertia of the cone, denoted I is obtained by summing (integrating) over all the disks that make up the cone, so that

$$I = \int_0^1 8\rho\pi x^4\,dx = \frac{8\rho\pi}{5}$$

The mass of the cone, denoted m, is its density times its volume. The volume of a cone is $\frac{1}{3}$(base area)(height). Hence $m = \frac{4}{3}\pi\rho$. Eliminating ρ gives $I = 6m/5$.
9. (i) $\frac{\pi^2}{2} + 2$, (ii) $\frac{13}{8}$

Spring 2003

1. (i) $y = \sqrt{\frac{2}{3}x^3 + 4x + d}$, (ii) $y = x + 2$, (iii) $y = \frac{x}{4} + \frac{c}{x^3}$.
2. 3.22 pm
3. (i) $y = e^{-x}(A\cos 4x + B\sin 4x)$, (ii) $y = \frac{7}{6}e^{x} - \frac{1}{2}e^{3x} + \frac{1}{3}e^{-2x}$.
4. (i) $\dfrac{\partial z}{\partial x} = 8x + 9x^2y^4$, $\dfrac{\partial z}{\partial y} = -1 + 12x^3y^3$.
(ii) $\dfrac{\partial z}{\partial x} = \dfrac{y^2 - x^2}{(x^2+y^2)^2}$, $\dfrac{\partial z}{\partial y} = -\dfrac{2xy}{(x^2+y^2)^2}$.
5. (i) $z_x = 2x - 2xy$, $z_y = 4y - x^2$, $z_{xx} = 2 - 2y$, $z_{yy} = 4$, $z_{xy} = -2x$.
(ii) $(0,0)$ minimum, $(2,1)$ saddle, $(-2,1)$ saddle.
6. (i) Rank $(A|\mathbf{b}) = 3$ and Rank $(A) = 2$, therefore no solution exists.
(ii) $x = -2$, $y = 3$, $z = 6$.
7. (i) 2
8. $(0, \pi/8)$
9. (i) 1, (ii) $2 - 2e^{-1/2}$.

Spring 2004

1. (i) $y = Ae^{-\sin x}$, (ii) $y = \sqrt{5e^{2(x-1)} - 1}$, (iii) $y = x^4 \ln x + cx^4$.
2. 9.54am.
3. (i) $y = e^{2x}(\cos 3x - \frac{2}{3}\sin 3x)$, (ii) $y = Ae^{x} + Be^{-3x} - \frac{1}{3}\sin 3x - \frac{1}{6}\cos 3x$.
4. (i) $z_x = 1 + y^2 + 8y^3x$, $z_y = 2xy + 12y^2x^2$,
(ii) $z_x = x^2y\cos xy + 2x\sin xy$, $z_y = x^3\cos xy$.
5. (i) $z_x = 3x^2 - 2y$, $z_y = -3y^2 - 2x$, $z_{xx} = 6x$, $z_{yy} = -6y$, $z_{xy} = -2$.
(ii) $(0,0)$ saddle, $(-\frac{2}{3}, \frac{2}{3})$ maximum.
6. $x = -2$, $y = 3$, $z = 2$.
7. $x = -12 + 3\alpha$, $y = 21 - 5\alpha$, $z = \alpha$.
8. (i) 2, (ii) -36.
9. $(\frac{3}{4}, \frac{8}{5})$.
10. (i) $\frac{1}{24}$, (ii) 18.

Spring 2005

1. (i) $y = \ln(\ln x + c)$, (ii) $y = (67 - 3\cos x)^{1/3}$, (iii) $y = \frac{1}{2} + \frac{c}{(x+1)^2}$
2. 12.46 pm
3. (i) $y = e^{-x}(A\cos 5x + B\sin 5x)$, (ii) $y = 2e^{-x} + 5e^{2x} + 3x - 4$
4. (i) $\frac{\partial z}{\partial x} = y + 12y^2x^3$, $\frac{\partial z}{\partial y} = 6y + x + 6yx^4$,
(ii) $\frac{\partial z}{\partial x} = \frac{2x^2+4xy}{(x+y)^2}$, $\frac{\partial z}{\partial y} = -\frac{2x^2}{(x+y)^2}$,
5. $x = y = 1.817$m, $z = 0.909$m.
6. (i) $x = 3$, $y = -5$, $z = 2$, (ii) no solution
7. (i) 2
8. $2M/3$
9. (i) $\frac{1}{8}$, (ii) $\frac{4}{3}$

Spring 2006

1. (i) $y = Ae^{-x^2/2}$, (ii) $y = \ln(\frac{1}{2} + \frac{1}{2}e^{2x})$, (iii) $y = -3x^2 + cx$.
2. 57.48^{o}F
3. (i) $y = e^{2x}(A\cos 3x + B\sin 3x)$
(ii) $y = \frac{1}{15}e^{2x} + \frac{5}{6}e^{-x} + \frac{1}{10}e^{-3x}$
4. (i) $\dfrac{\partial z}{\partial x} = 2xy + 4y^3$, $\dfrac{\partial z}{\partial y} = x^2 + 12xy^2$,
(ii) $\dfrac{\partial z}{\partial x} = \dfrac{x}{x + \sin y} + \ln(x + \sin y)$, $\dfrac{\partial z}{\partial y} = \dfrac{x\cos y}{x + \sin y}$
5. (i) $z_x = 3y - 3x^2$, $z_y = 3x - 3y^2$, $z_{xx} = -6x$, $z_{yy} = -6y$,
$z_{xy} = 3$
(ii) $(0,0)$ saddle, $(1,1)$ maximum
6. $x = -7$, $y = -11$, $z = -15$.
7. $A^2 = \begin{pmatrix} 17 & 10 & 14 \\ 10 & 8 & 4 \\ 14 & 4 & 20 \end{pmatrix} \qquad A^3 = \begin{pmatrix} 99 & 54 & 90 \\ 54 & 36 & 36 \\ 90 & 36 & 108 \end{pmatrix}$.

8. (i) 2
9. (i) $\frac{1}{4}$, (ii) $\frac{13}{210}$
10. $(\frac{2}{3}, \frac{4}{3})$.

Spring 2007

1. (i) $y = \frac{1}{1-x^3}$ (ii) $y = Ae^{\sin x} - 1$ (iii) $y = -x^3e^{-x} - x^2e^{-x} + cx^2$
2. $V = \left(\dfrac{3V_0^{1/3} - kt}{3}\right)^3$
3. (i) $y = \frac{11}{9}e^{-5x} + \frac{16}{9}e^{4x}$,
(ii) $y = e^{-x}(A\cos 2x + B\sin 2x) + \frac{3}{5}\cos x + \frac{3}{10}\sin x$
4. (i) $\frac{\partial z}{\partial x} = 2 + 2xy^4 + y^3$, $\frac{\partial z}{\partial y} = 1 + 4x^2y^3 + 3xy^2$
(ii) $\frac{\partial z}{\partial x} = 2xy^2\cos(x^2y)$, $\frac{\partial z}{\partial y} = \sin(x^2y) + x^2y\cos(x^2y)$.
5. $z_x = 2x - 6y + 3$, $z_y = 3y^2 - 6x + 6$, $z_{xx} = 2$, $z_{yy} = 6y$, $z_{xy} = -6$.
stationary points: $(\frac{27}{2}, 5)$ minimum; $(\frac{3}{2}, 1)$ saddle.
6. $x = -3$, $y = 1$, $z = 4$.
7. Condition is Rank $A =$ Rank $A|\mathbf{b}$.
System is solvable when $c + b = a$.
8. (i) 2, (ii) $x = 1$ or -3.
9. (i) $\frac{1}{6}$, (ii) $-\frac{8}{15}$.
10. $8M/3$.

Spring 2008

1. (i) $y = Ae^{\sin x}$, (ii) $y = -\sqrt{2x - 2x^2 + 4}$, the negative square root must be chosen because of the need to satisfy $y(1) = -2$, (iii) $y = \frac{2}{3}(x+1) + c/(x+1)^2$.
2. $c = \dfrac{p}{p+r}(1 - e^{-(p+r)t/V})$, $\lim_{t\to\infty} c(t) = p/(p+r)$.
3. (i) $y = \frac{5}{7}e^{-4x} + \frac{2}{7}e^{3x}$,
(ii) $y = e^{-x}(A\cos x + B\sin x) + \frac{1}{2}x^2 - \frac{3}{2}x + 1$.
4. (i) $\frac{\partial z}{\partial x} = 10y - 7y^2 - 3x^2y + 3$, $\frac{\partial z}{\partial y} = 10x - 14xy - x^3 - 6$.
(ii) $\frac{\partial z}{\partial x} = y + \dfrac{e^x}{y+1}$, $\frac{\partial z}{\partial y} = x - \dfrac{e^x}{(y+1)^2}$.
5. $x = y = 1.442$, $z = 2.884$.
6. $x = \frac{1}{2}$, $y = \frac{3}{2}$, $z = -1$.
7. $A^2 = \begin{pmatrix} 9 & 10 & 1 \\ 9 & 12 & 0 \\ 6 & 7 & 1 \end{pmatrix}$, $A^3 = \begin{pmatrix} 39 & 49 & 1 \\ 42 & 51 & 3 \\ 27 & 34 & 1 \end{pmatrix}$.
8. (i) 2
9. (i) $\frac{200}{3}$, (ii) $\frac{64}{5\pi}$.
10. $(\bar{x}, \bar{y}) = (\pi/2, \pi/8)$. Note that $\bar{x} = \pi/2$ comes directly from the symmetry of the graph about the line $x = \pi/2$.

Spring 2009

1. (i) $y = \ln(e^{2x} + e - 1)$, (ii) $y = (Ae^{\frac{2}{3}\sin 3x} - 1)/2$, (iii) $y = 3x/2 + c/x^3$.
2. $v = \dfrac{4v_0}{(2 + kt\sqrt{v_0})^2}$.
3. (i) $y = 3\cos 4x - \frac{1}{2}\sin 4x$, (ii) $y = Ae^{3x} + Be^{-5x} + \frac{2}{9}e^{4x}$.
4. (i) $z_x = 8x^3 + 3y^5 - 6xy^3$, $z_y = 15xy^4 - 9x^2y^2 + 8$,
(ii) $z_x = -2xy\sin(x^2 + y)$, $z_y = \cos(x^2 + y) - y\sin(x^2 + y)$.
5. (i) $z_x = -3x^2 + 12x + 6y - 12$, $z_y = -3y^2 + 6x - 12$, $z_{xx} = -6x + 12$, $z_{yy} = -6y$, $z_{xy} = 6$.
(ii) $(2, 0)$ saddle, $(4, 2)$ maximum
6. $x = -\frac{1}{2}$, $y = \frac{1}{2}$, $z = 1$.
7. They will have solutions other than $(0, 0, 0)$ when the determinant of the coefficient matrix is zero. This happens when $k = 1$. You can reach this conclusion by row reduction as well.
8. (i) 3
9. (i) $\frac{4}{3}$, (ii) $\frac{1}{2}e - 1$.
10. $3\rho\pi^2/16$.